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Donald J. Smith

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Donald J. Smith
Associate Professor of Finance and Economics
Boston University School of Management
595 Commonwealth Avenue
Boston, MA 02215

Phone: 617-353-2037
Fax: 617-353-6667
Email: donsmith@bu.edu

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Abstract

This paper analyzes the before-tax and after-tax cash flows for two stylized types of inflation-indexed bonds—"P-Linkers" and "C-Linkers." P-Linkers, like U.S. TIPS, have a fixed interest rate and link the accrued principal to changes in the consumer price index. C-Linkers are floating-rate notes that adjust the coupon interest rate for inflation while the principal is held constant. While both types of linkers provide protection from unexpected inflation, they differ significantly in terms of the amount and timing of cash flows, how and when cash flows are taxed, as well as in their price sensitivities to changes in real rates (i.e., their duration statistics). These differences impact investor strategies, especially those at aim to match the durations of assets and liabilities, and decisions about holding the inflation-indexed bonds in a tax-deferred structure like a retirement plan.

Alternative Designs for Inflation-Indexed Bonds: P-Linkers vs. C-Linkers

The market for inflation-indexed bonds, often called "linkers" because their cash flows are linked to changes in the consumer price index, has grown enormously in recent years. The history of linkers goes back to 1780, when the Commonwealth of Massachusetts issued notes tied to the prices of corn, beef, wool and leather; see Shiller [2003]. Many countries now issue inflation-linked debt, including the United Kingdom starting in 1981, Australia in 1985, Canada in 1991, and the United States in 1997. Price [1997] and Deacon, Derry, and Mirfenereski [2004] provide thorough histories and lists of international issuers and descriptions of the specific inflation-indexed securities.

Economists have long been interested in the effects of inflation indexation on monetary and fiscal policy and on retirement savings. If nothing else, the difference in pricing on traditional fixed-income bonds and linkers provides a market-based estimate of inflationary expectations. Shiller [2003] recounts that a standing joke at the U.S. Treasury Department was that when inflation-indexed bonds were to be issued, the prospectus should be sent first to the roster of the American Economics Association, as these would be the likely buyers. Fortunately, the demand for linkers has grown far beyond that investor base.

There are two primary types of inflation-indexed bonds: on the first type the coupon interest rate remains constant over the lifetime of the bond while the principal is adjusted each period for changes in the consumer price index, and on the second type the principal remains constant as the coupon rate is adjusted. The first type can be called "P-Linkers" (the link is to the principal) and the second "C-Linkers" (the link is to the coupon rate). P-Linkers are usually issued by governments, including U.S. Treasury TIPS, the acronym for Treasury Inflation-Protected Securities.[1] C-Linkers tend to be issued by commercial banks, life insurance companies, or federal agencies such as Fannie Mae. Rosen, Schumacher, and Casaudoumecq [1999] illustrate how companies issuing C-Linkers can convert to more traditional debt structures using swap contracts.

The objective of this paper is to analyze the differences between stylized versions of P-Linkers and C-Linkers in terms of the amount and timing of coupon interest and principal redemption cash flows, both before and after taxes, as well as their respective price sensitivities (i.e., duration statistics) to changes in real interest rates and inflation rates. The actual securities are necessarily complex because of their link to arbitrarily measured and reported price indices. The "indexation lag", the time between measuring the price index and adjusting the cash flow on the bond, is an obvious problem that will be neglected here. Tax effects will be stylized as well by assuming that the income tax rate is constant and tax obligations are paid when cash flows are received by the investor. Also, the numerical examples will assume an annual coupon payment whereas in practice payments are more frequent, for example, semi-annually on TIPS and monthly on many C-Linkers. This analysis of alternative designs for inflation-indexed bonds builds off of and extends earlier work by Babbel [1984] and Roll [1996].

Before-Tax Cash Flows on P-Linkers and C-Linkers

Exhibit 1 illustrates the cash flows on a 10-year, 2.50% P-Linker assuming an arbitrary sequence of low inflation rates in Panel A and high inflation rates in Panel B. The CPI is 100 on date 0 and in the first case rises to 130.199 for a compounded annual inflation rate of 2.674%, and in the second case rises to 302.977 for an average inflation rate of 11.723%. Notice that the accrued principal starts at a par value of $1,000 and then is adjusted each year for the percentage change in the CPI.

Each interest payment in nominal terms (i.e., before inflation) is the fixed coupon rate of 2.50% times the accrued principal. The final cash flow on date 10 includes the redemption of the entire accrued principal along with the last interest payment. P-Linkers typically provide protection against deflation as well as inflation. The interest payment each period will reflect the current accrued principal, which could be lower than at issuance due to falling price levels. However, deflation protection means that at maturity the investor will receive the original par value or the accrued principal, whichever is higher.

The real value of each cash flow is the nominal payment deflated by the CPI for that date. Panels A and B in Exhibit 1 demonstrate the salient feature to a P-Linker—the real cash flows are the same whether inflation turns out to be high or low. In real terms each coupon payment is worth a consistent $25 per $1,000 in par value for all changes in the level of consumer prices.

The internal rates of return (IRRs) on the nominal cash flows and the real values are calculated for three initial prices for the P-Linker: $980, $1,000, and $1,020. The realized IRR on the nominal flows will be higher when the subsequent path for the CPI is higher and if the security is initially purchased at a discount rather than at a premium. The key point is that an investor is able to lock in a real rate of return given the purchase price regardless of the CPI path and thereby is protected from inflation risk. If the P-Linker is purchased for $980, the real IRR is 2.731% in both panels. If purchased for

$1,020, it is 2.274% for both high and low inflation. If the initial price is $1,000, the realized IRR matches the 2.50% fixed coupon rate.

The average life statistic in Exhibit 1 is a crude measure of the "timeliness" of the cash flows. It is simply the weighted average of the dates of the cash flows whereby the weights are the shares of total cash received by the investor. Unlike duration, which will be discussed later, the time value of money is not considered. In the low inflation example in Panel A, the average life is 9.22 years for the nominal cash flows and 9.10 years for the real values. The results in Panel B for high inflation show that the average life of the nominal cash flows is higher, 9.51 years, while the average life for the real values remains at 9.10 years. In sum, the timing of the real cash flows as well as their IRR is independent of the future path to the consumer price index.

Exhibit 2 displays the nominal and real cash flows for a 10-year, 2.50% C-Linker for the same sequences of low and high inflations rates as in Exhibit 1. The C-Linker is a floating-rate note for which the nominal interest rate for each year is the realized inflation rate plus a fixed component, here 2.50%. The principal remains the same and is redeemed in full at maturity. C-Linkers protect against deflation in that the coupon payment is always non-negative. If the drop in the CPI is more than 2.50%, the coupon rate will be no lower than zero.

Notice that the cash flows are more "front-loaded" than on the P-Linker. The average life is 8.46 years in the low inflation example in Panel A of Exhibit 2 and down to 7.38 years with high inflation in Panel B. The realized inflation rate has a significant impact on the timing of the C-Linker's cash flows. Moreover, higher inflation lowers the average life of the C-Linker whereas it raises the average life of the P-Linker.

The IRRs of the nominal cash flows and the real values for the C-Linker in Exhibit 2 are lower than the IRRs for the P-Linker in Exhibit 1 for each assumed purchase price and each future path for the CPI. For instance, even when the 2.50% C-Linker is assumed to be purchased at par value for $1,000, the realized real rate of return is less than 2.50%. Moreover, the higher the inflation rate, the lower is the realized real rate of return. It is 2.435% in Panel A and only 2.239% in Panel B.

The reason for this underperformance is inherent in the design of the C-Linker. The problem is that its interest payment formula does not include the cross-product between the real rate and the inflation rate. The standard relationship between nominal, real, and inflation rates is: (1 + nominal rate) = (1 + real rate) * (1 + inflation rate). This can be rewritten as: nominal rate = real rate + inflation rate + (real rate * inflation rate). The last term is the cross-product. The C-Linker pays the investor only the first two terms. Simply put, the C-Linker does not compensate the investor for the full effect of year-to-year increases in the CPI. If the same issuer offered otherwise comparable C-Linkers and P-Linkers (i.e., same fixed rate, same maturity, same payment frequency), the C-Linker obviously would trade at a lower price than the P-Linker to reflect the deficiency in the interest payment formula.

Exhibit 3 shows the cash flows on a re-designed 10-year, 2.50% inflation-indexed note that includes the cross-product term—it is denoted C*-Linker. The principal is again fixed but the interest payment each year is: 2.50% + inflation rate + (2.50% * inflation rate). Obviously, all the cash flows will be higher than in Exhibit 2. The timing of cash flows as measured by the average lives is only slightly changed. Notice that now the 2.50% C*-Linker purchased at par value has a real IRR of 2.50% for both high and low inflation. It will deliver a locked-in real rate of return like the P-Linker independent of the subsequent path for the consumer price index.

Unfortunately, the equivalence in the real rates of return between the P-Linker and C*-Linker only holds if the notes are priced at par value. If the C*-Linker is purchased initially at a discounted price of $980, the IRR on the real values is 2.759% for low inflation and 2.854% for high inflation. If the notes trade at a premium and cost $1,020, the comparable real rates of return are 2.248% and 2.156%. Therefore, including the cross-product term in the design of an interest-indexed linker is an improvement but still is in general unable to lock in a pre-tax real rate of return independent of inflation—only the P-Linker does that.

While in practice most floating-rate notes tied to inflation rates are C-Linkers, C*-Linkers actually exist. Deacon, Derry, and Mirfenereski [2004] refer to them as "current pay bonds" and identify the government of Turkey as the only issuer in their survey. Roll [1996] reports that the U.S. Treasury considered a floating-rate note tied to changes in the CPI before settling on the P-Linker design. His description makes it clear that these would have been C*-Linkers rather than C-Linkers because the cross-product term would be included. The Treasury, however, did choose to structure Series I Saving Bonds (called I-Bonds) as C*-Linkers when they were first issued in 1998. These retail-oriented, non-marketable, 30-year linkers can be purchased by investors for small denominations, as low as $25. The coupon interest rate is reset semi-annually each May and November based on changes in the CPI. Interest accrues over the lifetime of the bond and is paid at redemption. As the I-Bonds are purchased at par value, the investor is able to lock in a real rate of return for either high or low future inflation rates.

After-Tax Cash Flows on P-Linkers and C-Linkers

Exhibits 4 and 5 illustrate the tax differences between P-Linkers and C-Linkers in the same high and low inflation environments as above. In each example the notes are assumed to be purchased at par value and held to maturity. That focuses attention to the taxation of ordinary income because there would not be any capital gain or loss to deal with. The tax rate is assumed to be 30% and the tax obligation is payable upon receipt of the interest payment.

Consider first the 2.50% P-Linker in the low inflation rates scenario of Panel A in Exhibit 4. On date 1, the investor receives $25.68 in interest income (the fixed 2.50% coupon rate times the accrued principal of $1,027) and owes $15.80 in taxes. That obligation is the sum of two parts: $7.70 for the tax on the interest income ($25.68 times 30%) and $8.10 for the tax on the increase in the accrued principal ($27 times 30%). This second part is perhaps surprising—the increase in the principal is taxable income each year even though the principal added due to inflation is not paid to the investor until maturity. This is known as "phantom income" and is how TIPS are taxed in the U.S.; in the U.K. the adjustment to the principal on their P-Linkers is not taxed.

Roll [1996] explains that taxation of the increase in accrued principal is intended to foster the liquidity of TIPS and to make them attractive to tax-exempt institutions like pension funds. On traditional Treasury bonds, expected inflation is part of the nominal interest rate and therefore is taxed. If TIPS did not similarly tax the accruals to principal due to inflation, they would have a tax benefit and trade at relatively higher prices and lower yields. Therefore, they would be avoided by pension funds, reducing market demand. Roll pointed to the illiquidity of P-Linkers in the U.K. at that time as evidence of this phenomenon.

The high inflation scenario in Panel B of Exhibit 4 reveals the potential cash flow problem facing the owner of the P-Linker on which the accrued principal is taxed. If the change in the CPI is high enough, the tax obligation can exceed the interest payment causing a negative after-tax cash flow. On date 1 the interest income is $27.70 (2.50% times $1,108) and the tax obligation is $40.71, the sum of $8.31 ($27.70 times 30%) and $32.40 (the increase in accrued principal of $108 times 30%). The after-tax cash flow to the investor is negative $13.01.

The threshold inflation rate for negative after-tax cash flows on the P-Linker in this example is 6.1947%. That is, if the percentage change in the CPI in any year exceeds that particular rate, the tax obligation exceeds the interest payment received. To determine this rate, let "AP_n" and "AP_{n-1}" denote the accrued principals for year n and year n – 1; "c" the fixed coupon rate on the P-Linker; "t" the tax rate; and "i" the inflation rate. Note that $i = (AP_n - AP_{n-1})/(AP_{n-1})$. The interest income for year n is $[c * AP_n]$ and the tax obligation is $[t * c * AP_n + t * (AP_n - AP_{n-1})]$. Equating these two expressions and solving for the inflation rate, denoted "i-threshold", yields this equation:

$$\text{i-threshold} = \frac{c * (1-t)}{t - c * (1-t)} \qquad (1)$$

Substituting c = 0.0250 and t = 0.30 into equation (1) gives the result that i-threshold = 0.061947. In general, the higher the fixed coupon rate and the lower the tax rate, the higher is the threshold level of inflation that results in a negative after-tax cash flow.

Exhibit 5 demonstrates the tax advantage of C-Linkers (and C*-Linkers as well) in relation to P-Linkers—the after-tax cash flows remain positive for all inflation rates.

The tax obligation each year is simply the tax rate times the interest income. For example, in Panel A on date 2 the coupon rate on the 2.50% C-Linker is shown to be 5.30%, the sum of the assumed inflation rate for the year of 2.80% plus 2.50%. The before-tax cash flow is $53.00 because the principal remains constant at $1,000. The tax obligation is $15.90 ($53.00 times 30%), leaving $37.10 after taxes. The U.S. Treasury Series I Savings Bonds, noted above to be C*-Linkers, allow the taxpayer to pay the tax on the interest income each year or defer it to the time when the bonds are redeemed.

Taxes obviously reduce the investor's rate of return. In the low inflation scenario and assuming the 30% tax rate, the IRR on the after-tax real values is only 0.969% for the P-Linker in Exhibit 4 and 0.922% for the C-Linker in Exhibit 5. Moreover, the after-tax real IRR becomes negative for both designs when the inflation rate is high enough. With the P-Linker, this happens when the average inflation rate exceeds the threshold rate calculated above. In Panel B of Exhibits 4 and 5 for high average inflation, the IRRs on the after-tax real values are minus 1.397% for the P-Linker and minus 1.571% on the C-Linker. A key point is that even the P-Linker, which as shown above can lock in a before-tax real rate of return independent of the purchase price and the subsequent inflation path, cannot lock in an after-tax real rate of return. That IRR will depend on future inflation rates because the increase in the accrued principal is treated as taxable income.

These after-tax returns explain why inflation-indexed bonds are usually held in tax-deferred structures such as defined-benefit pension funds, 401(k) and 403(b) plans, and individual retirement accounts. These allow the investment to grow at the higher pre-tax yield. There is another tax angle in the U.S. to investing with government-issued

TIPS. Interest income on Treasury securities is exempt from state and local government income tax. To the extent that this tax advantage is factored into market prices, the yield is reduced relative to otherwise comparable fully taxed debt securities. Usually an investor would avoid placing tax-exempt bonds, such as municipal securities that are exempt from federal income tax, in a tax-deferred structure. Therefore, holding TIPS in a defined benefit or defined contribution pension plan entails paying for an unneeded tax exemption.

Price Sensitivities of P-Linkers and C-Linkers

The impact on the price of a security resulting from a change in market interest rates is commonly summarized by a duration statistic. The duration of a traditional fixed-income bond estimates the percentage change in the price given a change in its yield to maturity. The general relationship is:

$$\frac{\Delta B}{B} \approx -D_y * \Delta y \tag{2}$$

whereby "B" is the price of the bond, "ΔB" is the change in the price, "D_y" is the duration statistic, and "Δy" is the change in the yield. To be precise, duration here is known as the "modified, or adjusted, duration" statistic.[2]

The change in the bond's yield to maturity can be due to either a change in the real rate or in the inflation rate. Changes in credit quality, liquidity, and tax status can also affect the yield but those factors are neglected here. A key point is that standard duration analysis does not distinguish between changes in the real rate and inflation. In effect, real rate duration and inflation duration statistics on a fixed-income bond are essentially the same number.[3] For example, a 6%, annual payment, 10-year bond has a duration statistic of 7.477 if its yield to maturity is 5.25%. If that yield were to drop by 10 basis points, either because of a change in the real rate or in the inflation rate (or some combination of the two), the estimated percentage increase in the bond price is 0.7477.

In contrast to traditional fixed-income bonds, inflation-indexed securities by design have quite different price sensitivities to changes in real rates compared to inflation rates. Their durations with respect to inflation should be close to or equal to zero. On the other hand, their real rate durations should be relatively high compared to fixed-income bonds having the same maturity, indicating substantial price volatility when real rates vary.

Equations for the durations for the stylized P-Linker, C-Linker, and C*-Linker with respect to changes in real rates and in inflation rates are derived formally in the Appendix and are summarized in Exhibit 6. While hardly obvious by inspection of these equations, the numerical examples to follow will illustrate that the inflation durations for the C-Linker and C*-Linker are close to zero and can even be slightly negative. The inflation duration for the P-Linker is always zero.

Exhibits 1, 2, and 3 show the cash flows for arbitrary realized sequences of low and high inflation rates. Now assume that those same inflation rates, 2.674% and 11.723%, are projected as of date 0. Exhibit 7 displays the real rate and inflation durations for the 10-year, 2.50% P-Linker, C-Linker, and C*-Linker corresponding once again to initial prices of $980, $1,000, and $1,020. In these calculations, the real rates are taken from the IRRs on the real cash flows in Exhibits 1, 2, and 3. The nominal rates, for each inflation assumption, are from the standard relationship: (1 + nominal rate) = (1 + real rate) * (1 + inflation rate).

It is perhaps surprising that the inflation duration on the C*-Linker becomes negative if the price is sufficiently below par value. Note that it is minus 0.081 if the price is $980 in Panel A and minus 0.062 in Panel B. This represents the rare circumstance whereby lower inflation, which lowers nominal yields and causes traditional fixed-income bond prices in general to rise, actually would lead to a slightly lower price for the inflation index-linked security.[4]

The phenomenon of negative inflation duration for C*-Linkers was first identified and discussed by Roll [1996]. The closed-form expressions for inflation duration derived in the Appendix can be manipulated algebraically to establish the conditions for this curious result. Using (A22), the sign of the inflation duration (D_i) for the C*-Linker reduces to:

$$Di \begin{matrix} > \\ = \\ < \end{matrix} 0 \quad \text{if} \quad (c-r) * \left[\frac{\left((1+n)^m - 1\right)}{n} - m \right] \begin{matrix} > \\ = \\ < \end{matrix} 0 \qquad (3)$$

Here "c" is the fixed coupon rate, "r" the real rate, "n" the nominal interest rate, and "m" the number of periods to maturity. The expression in brackets is positive whenever there is more than one period remaining (m > 1). From (A18), the C*-Linker is priced at a premium when c > r and at a discount when c < r. Therefore, the sign of the inflation duration statistic is positive when the floater trades at a premium above par value and are negative when it is priced at a discount below par value. The inflation duration is zero if the floater is at par value (c = r) or if there is just one period remaining (m = 1).

To investors, a significant observation seen in these numerical examples is that duration statistics for the C-Linkers depend critically on the projected inflation rate. Only the P-linker has the same real rate duration for both inflation scenarios. The 2.50% C-

Linker purchased at par value has a real rate duration of 7.862 if the percentage changes in the CPI are expected to be low on average (i.e., 2.674%). However, the real rate duration is significantly lower at 5.777 if inflation is expected to average 11.723%. The same pattern holds with the C*-Linkers—the higher the inflation rate, the lower the real rate duration.

This effect is exacerbated with increasing time to maturity. Consider much longer times to maturity: 30-year, 2.50% P-Linkers and C*-Linkers. If the real rate is 2.50%, both will be priced at par value and have inflation durations of zero. Using the formulas in Exhibit 6, the P-Linker has a real rate duration statistic of 20.930 regardless of the inflation rate. At a projected inflation rate of 2.764% over the 30 years, the C*-Linker has a real rate duration of 15.359. However, at 11.723% inflation, the real rate duration is considerably lower, only 7.565.

The sensitivity of the C-Linker's (and C*-Linker's) real rate duration to the projected inflation rate is particularly relevant to an investment strategy that aims to match the duration of assets to liabilities. Laurence Siegel and M. Barton Waring [2004, 2007] promote such a strategy for defined-benefit pension funds. The idea is that the level of retirement benefits typically is sensitive to changes in both real rates and inflation but to varying degrees. Usually, the real rate duration of the liabilities is much higher than the inflation duration. Immunizing the interest rate risk with traditional fixed-income bonds is problematical because these securities, as noted above, have essentially the same durations for real rates and inflation rates. Linkers having significantly different durations for each—a property Siegel and Waring dub "dual duration"—possess desirable risk-reducing attributes and, therefore, belong in the asset allocation of the pension fund.

P-Linkers have obvious advantages over C-Linkers for institutional investors employing immunization strategies that identify separate real rate and inflation duration targets. This point was first raised by Babbel [1984]. While all index-linked bonds have very low or zero inflation durations, only P-linkers have real rate durations that are independent of the inflation rate. As seen in the formulas and numerical examples, C-Linkers (and C*-Linkers) have real rate durations that depend critically on projected inflation. Therefore, an asset manager frequently might have to rebalance the portfolio to maintain the target duration as the assumed future inflation rate changes. Also, in periods of high inflation the real rate durations of even long-term C-Linkers might not be high enough to match the target for the liability.

Conclusions

Inflation-indexed securities are already an important asset class and are likely to grow in importance in coming years, especially if inflation expectations become more uncertain. While P-Linkers such as TIPS are the predominant design, C-Linkers are available in the market. Both types deliver protection from unexpected inflation. That is, their inflation durations are close to zero, meaning that a change in the projected track for future price levels has a minimal impact on the current price of the security, and their real rates of return over the time to maturity are relatively stable despite changes in the inflation rate.

There are significant differences between the two designs, however. The cash flows on C-Linkers are much more "front-loaded" than on P-Linkers and have a shorter average life, the more so the higher the inflation rate. That is an advantage if credit risk is a concern. P-Linkers offer a higher real rate duration that also is independent of the inflation rate. That is especially attractive to institutional investors that have identified a target duration with respect to real rates that differs from their target duration for inflation rates. C-Linkers, on the other hand, have lower real rate durations that depend critically on the projected path for future inflation. P-Linkers have a significant tax disadvantage relative to C-Linkers because the investor can experience negative cash flows if the inflation rate reaches a certain threshold.

There is ample room for further innovation in the design of linkers. Because of their tax disadvantages, P-Linkers like TIPS in the U.S. are mostly held within a segmented market—pension funds and individual retirement accounts. Given that, the ideal P-Linker for retirement savings would be long-term, fully taxable, and free of default risk. Unfortunately, at present the U.S. Treasury does not issue TIPS beyond 20 years to maturity, nor does it issue fully taxable debt. Investors not wanting or able to hold securities in a tax-deferred structure can buy C-Linkers, but these entail bearing corporate or agency credit risk and have real rate durations that vary with the projected inflation rate.

Ideally, governments should issue a variety of inflation-indexed bonds, including both P-Linkers and C-Linkers, and offer a full range of maturities (and real rate durations).

Appendix: Derivation of the Duration Statistics

for the P-Linker, C-Linker, and C*-Linker

Notation: m = number of periods to maturity

c = fixed coupon rate

F = initial face (or par) value

n = nominal interest rate

r = real interest rate

i = inflation rate

P = price of P-Linker

C = price of the C-Linker

C* = price of the C*-Linker

D_r = real rate duration

D_i = inflation duration

P-Linker

Given the inflation rate, the path for the accrued principal on the P-Linker will be: $(1 + i) * F, (1 + i)^2 * F, \ldots, (1 + i)^m * F$. The price of the P-Linker on date 0 is the present value of the interest and principal redemption cash flows, discounted at the nominal interest rate.

$$P = \left[\frac{c*(1+i)*F}{(1+n)} + \frac{c*(1+i)^2*F}{(1+n)^2} + \ldots + \frac{c*(1+i)^m*F}{(1+n)^m} \right] + \frac{(1+i)^m*F}{(1+n)^m} \quad \text{(A1)}$$

Use the customary relationship between nominal, real, and inflation rates. i.e., that (1 + n) = (1 + r) * (1 + i), to reduce this expression to:

$$P = \left[\frac{c*F}{(1+r)} + \frac{c*F}{(1+r)^2} + \ldots + \frac{c*F}{(1+r)^m} \right] + \frac{F}{(1+r)^m} \tag{A2}$$

The term in brackets is the sum of a finite geometric series. Reducing that term allows for a closed-form equation for the price of the P-Linker.

$$P = \frac{c*F}{r} * \left(1 - \frac{1}{(1+r)^m} \right) + \frac{F}{(1+r)^m} \tag{A3}$$

Some algebraic manipulation of (A3) results in the following:

$$\frac{P}{F} = 1 + \frac{c-r}{r} * \left(1 - \frac{1}{(1+r)^m} \right) \tag{A4}$$

The P-Linker will be priced at a premium (P > F) if the fixed coupon rate exceeds the real rate (c > r), at a discount (P < F) if c < r, and at par value (P = F) if c = r.

Define the (modified) duration statistics for changes in the real interest rate and in the inflation rate to be the following:

$$Dr \equiv -\frac{dP}{dr} * \frac{1}{P} \tag{A5}$$

$$Di \equiv -\frac{dP}{di} * \frac{1}{P} \tag{A6}$$

The inflation duration for the P-Linker is zero because the inflation rate has dropped out of the pricing equation in (A3) and, therefore, the derivative with respect to changes in the inflation rate is zero. This illustrates the key feature of the stylized P-Linker—its price at any time is insensitive to changes in the inflation rate.

The expression for the real rate duration of the P-Linker follows from the first derivative of the price in (A3) with respect to the real rate.

$$\frac{dP}{dr} = -\frac{c * F}{r^2} * \left(1 - \frac{1}{(1+r)^m}\right) + \frac{c * F}{r} * \left(\frac{m}{(1+r)^{m+1}}\right) - \frac{m * F}{(1+r)^{m+1}} \tag{A7}$$

Equations (A7) and (A3) are substituted into the expression for the real rate duration in (A5) and simplified algebraically.

$$Dr = \frac{1}{r} - \frac{1 + \left(\frac{m * (c - r)}{1 + r}\right)}{c * \left((1+r)^m - 1\right) + r} \tag{A8}$$

C-Linker

The price of a C-Linker that matures in m periods and makes an interest payment each period equal to the inflation rate plus a fixed rate times the constant face value can be expressed as:

$$C = \left[\frac{(i + c) * F}{(1 + n)} + \frac{(i + c) * F}{(1+n)^2} + \ldots + \frac{(i + c) * F}{(1+n)^m}\right] + \frac{F}{(1+n)^m} \tag{A9}$$

This reduces to the following closed-form expression:

$$C = \frac{(i + c) * F}{n} * \left(1 - \frac{1}{(1+n)^m}\right) + \frac{F}{(1+n)^m} \tag{A10}$$

This can be rearranged algebraically to identify when the C-Linker will be priced at a premium or discount.

$$\frac{C}{F} = 1 + \frac{c - r * (1 + i)}{n} * \left(1 - \frac{1}{(1+n)^m}\right) \tag{A11}$$

The price of the C-Linker will be above, equal to, and below par value if c > r * (1 + i), c = r * (1 + i), and c < r * (1 + i), respectively.

The duration statistics for the C-Linker will parallel those defined in (A5) and (A6) and will depend on the first derivatives of the pricing equation in (A10) with respect to changes in the real rate and in the inflation rate. These derivatives are:

$$\frac{dC}{dr} = -\frac{(1+i)*(c+i)*F}{n^2}*\left(1-\frac{1}{(1+n)^m}\right) + \frac{(c+i)*F}{n}*\left(\frac{m*(1+i)}{(1+n)^{m+1}}\right) - \frac{m*(1+i)*F}{(1+n)^{m+1}} \qquad \text{(A12)}$$

$$\frac{dC}{di} = -\frac{(1+r)*(c+i)*F}{n^2}*\left(1-\frac{1}{(1+n)^m}\right) + \frac{(c+i)*F}{n}*\left(\frac{m*(1+r)}{(1+n)^{m+1}}\right) + \frac{F}{n}*\left(1-\frac{1}{(1+n)^m}\right) - \frac{m*(1+r)*F}{(1+n)^{m+1}} \qquad \text{(A13)}$$

Substituting (A12) and (A13), along with (A10), into expressions for the definitions for the real rate and inflation durations of the C-Linker and simplifying via algebra give these results:

$$Dr = \frac{1+i}{n} - \frac{1+i+\left(\frac{m*\left(c-r*(1+i)\right)}{1+r}\right)}{(c+i)*\left((1+n)^m-1\right)+n} \qquad \text{(A14)}$$

$$Di = \frac{1+r}{n} - \frac{1+r+\left(\frac{m*\left(c-r*(1+i)\right)}{1+i}\right)}{(c+i)*\left((1+n)^m-1\right)+n} - \frac{(1+n)^m-1}{(c+i)*\left((1+n)^m-1\right)+n} \qquad \text{(A15)}$$

C*-Linker

The duration statistics for the C*-Linker can be derived in the same manner. Here the cross-product term (c * i) is included in the interest payment cash flows in the basic pricing equation.

$$C^* = \left[\frac{(i + c + i * c) * F}{(1+n)} + \frac{(i + c + i * c) * F}{(1+n)^2} + \ldots + \frac{(i + c + i * c) * F}{(1+n)^m} \right] + \frac{F}{(1+n)^m} \qquad \text{(A16)}$$

Its closed-form expression is:

$$C^* = \frac{(i + c + i * c) * F}{n} * \left(1 - \frac{1}{(1+n)^m} \right) + \frac{F}{(1+n)^m} \qquad \text{(A17)}$$

This can be arranged as:

$$\frac{C^*}{F} = 1 + \frac{(c - r) * (1 + i)}{n} * \left(1 - \frac{1}{(1+n)^m} \right) \qquad \text{(A18)}$$

The C*-Linker will be priced at a premium if c > r, at a discount if c < r, and at par value if c = r. Also, the amount of the premium or discount will be a function of the inflation rate—the higher the inflation rate, the larger the premium or discount that results when the fixed component of the coupon rate does not equal the real rate.

The first derivatives with respect to the real rate and the inflation rate are:

$$\begin{aligned} \frac{dC^*}{dr} = &- \frac{(1 + i) * (c + i + i * c) * F}{n^2} * \left(1 - \frac{1}{(1+n)^m} \right) \\ &+ \frac{(c + i + i * c) * F}{n} * \left(\frac{m * (1 + i)}{(1+n)^{m+1}} \right) - \frac{m * (1 + i) * F}{(1+n)^{m+1}} \end{aligned} \qquad \text{(A19)}$$

$$\frac{dC^*}{di} = -\frac{(1+r)*(c+i+i*c)*F}{n^2}*\left(1-\frac{1}{(1+n)^m}\right)$$

$$+\frac{(c+i+i*c)*F}{n}*\left(\frac{m*(1+r)}{(1+n)^{m+1}}\right)+\frac{(1+c)*F}{n}*\left(1-\frac{1}{(1+n)^m}\right) \quad \text{(A20)}$$

$$-\frac{m*(1+r)*F}{(1+n)^{m+1}}$$

The duration equations for the C*-Linker are:

$$Dr = \frac{1+i}{n} - \frac{1+i+\left(\frac{m*(c-r)*(1+i)}{1+r}\right)}{(c+i+i*c)*\left((1+n)^m-1\right)+n} \quad \text{(A21)}$$

$$Di = \frac{1+r}{n} - \frac{1+r+\left(m*(c-r)\right)}{(c+i+i*c)*\left((1+n)^m-1\right)+n}$$

$$-\frac{(1+c)*\left((1+n)^m-1\right)}{(c+i+i*c)*\left((1+n)^m-1\right)+n} \quad \text{(A22)}$$

End-Notes

1. P-Linkers are not the exclusive domain of governments. Tesco, the British supermarket retailer, issued a 36-year principal-indexed linker in March of 2006 (Financial Times, March 17, 2006).

2. The classic reference on bond duration is Macaulay [1938]. The "Macaulay duration" of a traditional fixed-income bond can be calculated as the weighted average of the times to receipt of cash flows, whereby the weights are shares of the total present value corresponding to each cash flow. "Modified, or adjusted, duration" is the Macaulay duration divided by one plus the yield to maturity.

3. The Macaulay real rate and inflation durations on a traditional fixed-income bond are the same. The modified real rate duration is the Macaulay duration divided by one plus the real rate; the modified inflation duration is the Macaulay duration divided by one plus the inflation rate. Therefore, the modified durations will be slightly different whenever the real and inflation rates differ.

4. Another circumstance whereby a floating-rate note can have negative duration occurs when the security trades at a substantial discount due to deterioration in credit quality. See Smith [2006].

References

Babbel, D. F. "Real Immunization with Indexed Bonds." *Financial Analysts Journal*, 40 November-December, (1984), pp. 49-54.

Deacon, M., A. Derry, and D. Mirfendereski. *Inflation-Indexed Securities: Bonds, Swaps, and Other Derivatives*, 2nd Edition, John Wiley & Sons, 2004.

Macaulay, F. R. *The Movements of Interest Rates, Bond Yields, and Stock Prices in the United States Since 1856*, National Bureau of Economic Research, 1938.

Price Robert, "The Rationale and Design of Inflation-Indexed Bonds," International Monetary Fund Working Paper No. 97/12, January 1997. (Available at the Social Science Research Network Paper Collection at http://ssrn.com/ abstract=882231)

Roll, R. "U.S. Treasury Inflation-Indexed Bonds: The Design of a New Security." *The Journal of Fixed Income*, December (1996), pp. 9-28.

Rosen, A., M. Schumacher, and J. Casaudoumecq. "Corporate and Agency Inflation-Linked Securities," Chapter 18 in John Brynjolfsson and Frank J. Fabozzi, ed., *Handbook of Inflation-Indexed Bonds*, Frank J. Fabbozi Associates, 1999.

Siegel, L. and M. B. Waring. "TIPS, the Dual Duration, and the Pension Plan." *Financial Analysts Journal*, 60 (2004), pp. 52-64.

Shiller, Robert J. "The Invention of Inflation-Indexed Bonds in Early America." Cowles Foundation, Yale University, Discussion Paper No. 1442, October 2003. (Available at the Social Science Research Network Paper Collection at http://ssrn.com/abstract =475881).

Smith, D. J. "Negative Duration: The Odd Case of GMAC's Floating-Rate Note." *Journal of Applied Finance* 16, Fall/Winter (2006), pp. 37-44.

Waring, M. B. and L. Siegel. "Don't Kill the Golden Goose! Saving Pension Plans." *Financial Analysts Journal*, 63, January/February (2007), pp. 31-45.

Exhibit 1: Before-Tax Cash Flows on a 2.50% P-Linker

Panel A: Low Inflation Rates

Date	CPI	Inflation Rate	Accrued Principal	Nominal Cash Flow	Real Value of Cash Flow
0	100.000		1,000.00		
1	102.700	2.70%	1,027.00	25.68	25.00
2	105.576	2.80%	1,055.76	26.39	25.00
3	109.007	3.25%	1,090.07	27.25	25.00
4	112.223	2.95%	1,122.23	28.06	25.00
5	115.084	2.55%	1,150.84	28.77	25.00
6	117.501	2.10%	1,175.01	29.38	25.00
7	119.616	1.80%	1,196.16	29.90	25.00
8	122.547	2.45%	1,225.47	30.64	25.00
9	126.100	2.90%	1,261.00	31.53	25.00
10	130.199	3.25%	1,301.99	1,334.54	1,025.00
Average Inflation Rate		2.674%	Average Life	9.22	9.10
			IRR at Price = 980	5.482%	2.731%
			IRR at Price = 1,000	5.244%	2.500%
			IRR at Price = 1,020	5.012%	2.274%

Panel B: High Inflation Rates

Date	CPI	Inflation Rate	Accrued Principal	Nominal Cash Flow	Real Value of Cash Flow
0	100.000		1,000.00		
1	110.800	10.80%	1,108.00	27.70	25.00
2	123.819	11.75%	1,238.19	30.95	25.00
3	140.287	13.30%	1,402.87	35.07	25.00
4	157.472	12.25%	1,574.72	39.37	25.00
5	173.849	10.40%	1,738.49	43.46	25.00
6	190.886	9.80%	1,908.86	47.72	25.00
7	212.170	11.15%	2,121.70	53.04	25.00
8	242.298	14.20%	2,422.98	60.57	25.00
9	272.707	12.55%	2,727.07	68.18	25.00
10	302.977	11.10%	3,029.77	3,105.52	1,025.00
Average Inflation Rate		11.723%	Average Life	9.51	9.10
			IRR at Price = 980	14.763%	2.731%
			IRR at Price = 1,000	14.505%	2.500%
			IRR at Price = 1,020	14.253%	2.274%

Exhibit 2: Before-Tax Cash Flows on a 2.50% C-Linker

Panel A: Low Inflation Rates

Date	CPI	Inflation Rate	Coupon Rate	Nominal Cash Flow	Real Value of Cash Flow
0	100.000				
1	102.700	2.70%	5.200%	52.00	50.63
2	105.576	2.80%	5.300%	53.00	50.20
3	109.007	3.25%	5.750%	57.50	52.75
4	112.223	2.95%	5.450%	54.50	48.56
5	115.084	2.55%	5.050%	50.50	43.88
6	117.501	2.10%	4.600%	46.00	39.15
7	119.616	1.80%	4.300%	43.00	35.95
8	122.547	2.45%	4.950%	49.50	40.39
9	126.100	2.90%	5.400%	54.00	42.82
10	130.199	3.25%	5.750%	1,057.50	812.22
Average Inflation Rate		2.674%	Average Life	8.46	8.25
			IRR at Price = 980	5.448%	2.693%
			IRR at Price = 1,000	5.183%	2.435%
			IRR at Price = 1,020	4.924%	2.183%

Panel B: High Inflation Rates

Date	CPI	Inflation Rate	Coupon Rate	Nominal Cash Flow	Real Value of Cash Flow
0	100.000				
1	110.800	10.80%	13.300%	133.00	120.04
2	123.819	11.75%	14.250%	142.50	115.09
3	140.287	13.30%	15.800%	158.00	112.63
4	157.472	12.25%	14.750%	147.50	93.67
5	173.849	10.40%	12.900%	129.00	74.20
6	190.886	9.80%	12.300%	123.00	64.44
7	212.170	11.15%	13.650%	136.50	64.34
8	242.298	14.20%	16.700%	167.00	68.92
9	272.707	12.55%	15.050%	150.50	55.19
10	302.977	11.10%	13.600%	1,136.00	374.95
Average Inflation Rate		11.723%	Average Life	7.38	6.18
			IRR at Price = 980	14.565%	2.589%
			IRR at Price = 1,000	14.174%	2.239%
			IRR at Price = 1,020	13.795%	1.899%

Exhibit 3: Before-Tax Cash Flows on a 2.50% C*-Linker

Panel A: Low Inflation Rates

Date	CPI	Inflation Rate	Coupon Rate	Nominal Cash Flow	Real Value of Cash Flow
0	100.000				
1	102.700	2.70%	5.268%	52.68	51.29
2	105.576	2.80%	5.370%	53.70	50.86
3	109.007	3.25%	5.831%	58.31	53.49
4	112.223	2.95%	5.524%	55.24	49.22
5	115.084	2.55%	5.114%	51.14	44.43
6	117.501	2.10%	4.653%	46.53	39.60
7	119.616	1.80%	4.345%	43.45	36.32
8	122.547	2.45%	5.011%	50.11	40.89
9	126.100	2.90%	5.473%	54.73	43.40
10	130.199	3.25%	5.831%	1,058.31	812.84
Average Inflation Rate		2.674%	Average Life	8.44	8.24
			IRR at Price = 980	5.516%	2.759%
			IRR at Price = 1,000	5.250%	2.500%
			IRR at Price = 1,020	4.991%	2.248%

Panel B: High Inflation Rates

Date	CPI	Inflation Rate	Coupon Rate	Nominal Cash Flow	Real Value of Cash Flow
0	100.000				
1	110.800	10.80%	13.570%	135.70	122.47
2	123.819	11.75%	14.544%	145.44	117.46
3	140.287	13.30%	16.133%	161.33	115.00
4	157.472	12.25%	15.056%	150.56	95.61
5	173.849	10.40%	13.160%	131.60	75.70
6	190.886	9.80%	12.545%	125.45	65.72
7	212.170	11.15%	13.929%	139.29	65.65
8	242.298	14.20%	17.055%	170.55	70.39
9	272.707	12.55%	15.364%	153.64	56.34
10	302.977	11.10%	13.878%	1,138.78	375.86
Average Inflation Rate		11.723%	Average Life	7.35	6.16
			IRR at Price = 980	14.860%	2.854%
			IRR at Price = 1,000	14.465%	2.500%
			IRR at Price = 1,020	14.082%	2.156%

Exhibit 4: After-Tax Cash Flows on a 2.50% P-Linker

30% Tax Rate

Panel A: Low Inflation Rates

Date	CPI	Inflation Rate	Accrued Principal	Before-Tax Cash Flow	Taxes Due	After-Tax Cash Flow	After-Tax Real Value
0	100.000		1,000.00	-1,000.00		-1,000.00	-1,000.00
1	102.700	2.70%	1,027.00	25.68	15.80	9.87	9.61
2	105.576	2.80%	1,055.76	26.39	16.54	9.85	9.33
3	109.007	3.25%	1,090.07	27.25	18.47	8.78	8.06
4	112.223	2.95%	1,122.23	28.06	18.06	9.99	8.90
5	115.084	2.55%	1,150.84	28.77	17.22	11.55	10.04
6	117.501	2.10%	1,175.01	29.38	16.06	13.31	11.33
7	119.616	1.80%	1,196.16	29.90	15.32	14.59	12.20
8	122.547	2.45%	1,225.47	30.64	17.98	12.65	10.33
9	126.100	2.90%	1,261.00	31.53	20.12	11.41	9.05
10	130.199	3.25%	1,301.99	1,334.54	22.06	1,312.48	1,008.06
					IRR	3.669%	0.969%

Panel B: High Inflation Rates

Date	CPI	Inflation Rate	Accrued Principal	Before-Tax Cash Flow	Taxes Due	After-Tax Cash Flow	After-Tax Real Value
0	100.000		1,000.00	-1,000.00		-1,000.00	-1,000.00
1	110.800	10.80%	1,108.00	27.70	40.71	-13.01	-11.74
2	123.819	11.75%	1,238.19	30.95	48.34	-17.39	-14.04
3	140.287	13.30%	1,402.87	35.07	59.93	-24.85	-17.72
4	157.472	12.25%	1,574.72	39.37	63.37	-24.00	-15.24
5	173.849	10.40%	1,738.49	43.46	62.17	-18.71	-10.76
6	190.886	9.80%	1,908.86	47.72	65.43	-17.71	-9.28
7	212.170	11.15%	2,121.70	53.04	79.76	-26.72	-12.59
8	242.298	14.20%	2,422.98	60.57	108.56	-47.98	-19.80
9	272.707	12.55%	2,727.07	68.18	111.68	-43.50	-15.95
10	302.977	11.10%	3,029.77	3,105.52	113.53	2,991.98	987.53
					IRR	10.165%	-1.397%

Exhibit 5: After-Tax Cash Flows on a 2.50% C-Linker

30% Tax Rate

Panel A: Low Inflation Rates

Date	CPI	Inflation Rate	Coupon Rate	Before-Tax Cash Flow	Taxes Due	After-Tax Cash Flow	After-Tax Real Value
0	100.000			-1,000.00		-1,000.00	-1,000.00
1	102.700	2.70%	5.20%	52.00	15.60	36.40	35.44
2	105.576	2.80%	5.30%	53.00	15.90	37.10	35.14
3	109.007	3.25%	5.75%	57.50	17.25	40.25	36.92
4	112.223	2.95%	5.45%	54.50	16.35	38.15	33.99
5	115.084	2.55%	5.05%	50.50	15.15	35.35	30.72
6	117.501	2.10%	4.60%	46.00	13.80	32.20	27.40
7	119.616	1.80%	4.30%	43.00	12.90	30.10	25.16
8	122.547	2.45%	4.95%	49.50	14.85	34.65	28.27
9	126.100	2.90%	5.40%	54.00	16.20	37.80	29.98
10	130.199	3.25%	5.75%	1,057.50	17.25	1,040.25	798.97
					IRR	3.626%	0.922%

Panel B: High Inflation Rates

Date	CPI	Inflation Rate	Coupon Rate	Before-Tax Cash Flow	Taxes Due	After-Tax Cash Flow	After-Tax Real Value
0	100.000			-1,000.00		-1,000.00	-1,000.00
1	110.800	10.80%	13.30%	133.00	39.90	93.10	84.03
2	123.819	11.75%	14.25%	142.50	42.75	99.75	80.56
3	140.287	13.30%	15.80%	158.00	47.40	110.60	78.84
4	157.472	12.25%	14.75%	147.50	44.25	103.25	65.57
5	173.849	10.40%	12.90%	129.00	38.70	90.30	51.94
6	190.886	9.80%	12.30%	123.00	36.90	86.10	45.11
7	212.170	11.15%	13.65%	136.50	40.95	95.55	45.03
8	242.298	14.20%	16.70%	167.00	50.10	116.90	48.25
9	272.707	12.55%	15.05%	150.50	45.15	105.35	38.63
10	302.977	11.10%	13.60%	1,136.00	40.80	1,095.20	361.48
					IRR	9.933%	-1.571%

Exhibit 6: Equations for the Modified Duration

of the P-Linker, C-Linker, and C*-Linker

Notation: D_r = real rate duration, D_i = inflation duration, m = maturity, c = fixed coupon rate, n = nominal interest rate, r = real interest rate, i = inflation rate.

Real Rate Modified Durations

P-Linker

$$Dr = \frac{1}{r} - \frac{1 + \left(\frac{m * (c - r)}{1 + r}\right)}{c * \left((1+r)^m - 1\right) + r}$$

C-Linker

$$Dr = \frac{1 + i}{n} - \frac{1 + i + \left(\frac{m * (c + i - n)}{1 + r}\right)}{(c + i) * \left((1+n)^m - 1\right) + n}$$

C*-Linker

$$Dr = \frac{1 + i}{n} - \frac{1 + i + \left(\frac{m * (c + i + i * c - n)}{1 + i}\right)}{(c + i + i * c) * \left((1+n)^m - 1\right) + n}$$

Inflation Rate Modified Durations

P-Linker

$$Di = 0$$

C-Linker

$$Di = \frac{1 + r}{n} - \frac{1 + r + \left(\frac{m * (c + i - n)}{1 + i}\right)}{(c + i) * \left((1+n)^m - 1\right) + n} - \frac{(1+n)^m - 1}{(c + i) * \left((1+n)^m - 1\right) + n}$$

C*-Linker

$$Di = \frac{1 + r}{n} - \frac{1 + r + \left(\frac{m * (c + i + i * c - n)}{1 + i}\right)}{(c + i + i * c) * \left((1+n)^m - 1\right) + n} - \frac{(1 + c) * \left((1+n)^m - 1\right)}{(c + i + i * c) * \left((1+n)^m - 1\right) + n}$$

Exhibit 7: Duration Statistics on

10-year, 2.50% P-Linkers, C-Linkers, and C*-Linkers

Price	**$980**	**$1,000**	**$1,020**
Panel A: Low Inflation, 2.674%			
P-Linker			
Real Rate Duration	8.720	8.752	8.783
Inflation Duration	0	0	0
C-Linker			
Real Rate Duration	7.820	7.862	7.903
Inflation Duration	0.107	0.186	0.263
C*-Linker			
Real Rate Duration	7.794	7.836	7.877
Inflation Duration	– 0.081	0.000	0.078
Panel B: High Inflation, 11.723%			
P-Linker			
Real Rate Duration	8.720	8.752	8.783
Inflation Duration	0	0	0
C-Linker			
Real Rate Duration	5.719	5.777	5.835
Inflation Duration	0.054	0.116	0.176
C*-Linker			
Real Rate Duration	5.653	5.712	5.770
Inflation Duration	– 0.062	0.000	0.060