Professor Andrew Hertzberg
Finance 430-71/72

Duration and Modified Duration

- **Goal:** Our goal here is to figure out how to quantify and control interest rate risk.
  - Interest rate movements are the *only* source of risk for default free bonds.
  - Here, we are worried about *price risk* as opposed to *reinvestment risk.*
    - Price Risk is the risk that, over the next day or month, the value of our portfolio will rise or fall as interest rates fall or rise.
  - **Duration** is our key tool for quantifying price risk.
- **Definition:** A bond's (or an investment's) duration (D) **is** defined as how long, from today, until you receive the bond (or investment) cashflows.
  - The duration of a zero coupon bond is the time, from today, until the maturity of the bond.
- **Definition:** A bond's **modified duration** (D^*) is its duration (D) divided by its yield-to-maturity:

$$D^* = \frac{D}{1+y}$$

- **Examples:**
  - What is the duration of a 10 year zero coupon bond?
    - *Answer:* $\boxed{10}$
  - 10 years ago you bought a 20 year zero coupon bond. What is its duration today?
    - *Answer:* $\boxed{10}$
  - What is the modified duration of a 10 year zero coupon bond if its yield-to-maturity is 5%?
    - *Answer:* $= 10/(1+0.05) = \boxed{9.52}$

Using Modified Duration to Calculate Price Responses to Interest Rate Shifts

- **Use:** A bond's (or an investment's) modified duration **tells you** approximately how much your bond or investment portfolio will change in value when interest rates move by 1%. Specifically:

$$\% \text{ Change in Bond Price} \approx -D^* \times \text{Change in Yield } (\%).$$

  - The minus sign in front of D^* reflects the fact that when the yield rises, the bond price falls.

- **Example:** A 20 year zero coupon bond with \$100 face value currently has a yield-to-maturity of 5%. If its yield falls to 4.9%, how much will its price change?

  - *Answer:*

$$D^* = \frac{20}{1.05} = 19.05$$

  So the price should change by approximately:

$$\begin{aligned} &= -D^* \times \text{Change in Yield } (\%) \\ &= -19.05 \times (-0.1\%) \\ &= \boxed{1.905\%} \end{aligned}$$

Calculating the Duration of a Portfolio

- We specified earlier that a bond's duration (D) **is** defined as how long, from today, until you receive the bond cashflows.
- For a zero coupon bond, which has only a single future cashflow, or some other investment with a single future cashflow, the duration is the time from today until you receive that cashflow.
- **Definition:** When there are multiple cashflows, the duration is the **value-weighted** average of when you receive the cashflows.
  - Specifically, we weight the durations of the cashflows by their present-values divided by the value of the portfolio, that is:

$$\begin{aligned}\text{Duration of Portfolio} &= \text{Duration}(\text{CF}_1) \times \left(\frac{\text{PV}(\text{CF}_1)}{\text{Portfolio Value}}\right) \\ &+ \text{Duration}(\text{CF}_2) \times \left(\frac{\text{PV}(\text{CF}_2)}{\text{Portfolio Value}}\right) \\ &+ \cdots\end{aligned}$$

Where the Portfolio Value is, as always, the just the sum of the PV's of the cashflows:

$$\text{Portfolio Value} = \text{PV}(\text{CF}_1) + \text{PV}(\text{CF}_2) + \cdots$$

Calculating the Duration of a Portfolio (2)

- **Example:** The term structure is flat at 5%. Consider a 2-year 8% coupon bond with face value \$100. What are this bond's duration and modified duration?

  Answer:

  – First, we need to recognize that this coupon bond is a portfolio of two cashflows.

  – This bond gives cashflows of \$8 in one year and \$108 in two years.

  – The bond value is currently the sum of the PV's of the two cashflows:

$$\begin{aligned} \text{Price} &= \frac{\$8}{1.05} + \frac{\$108}{(1.05)^2} \\ &= \$7.62 + \$97.96 \\ &= \$105.58 \end{aligned}$$

  – The duration of the two cashflows are 1 year and 2 years, respectively.

  – To get the duration of this portfolio we weight the durations by the present value of the cashflows, divided by the bond value:

$$\begin{aligned} D &= (1 \text{ year}) \times \left(\frac{\$7.62}{\$105.58}\right) + (2 \text{ years}) \times \left(\frac{\$97.96}{\$105.58}\right) \\ &= \boxed{1.9278} \\ D^* &= \frac{1.9278}{(1+0.05)} \\ &= \boxed{1.836} \end{aligned}$$

  As a reality check, we see that this is slightly less than 2 years, as it must be.

  ∗ Why? Most of the bond's value comes from the \$108 we receive in 2 years. The rest of the value is due to the \$8 we receive in 1 year.

Calculating the Duration of a Portfolio (3)

- **Example:** The term structure is flat at 5%. Consider a 20-year 5% coupon bond with face value \$1000. Find this bond's duration, and modified duration, and calculate how much the price of this bond would fall (in %) if its yield fell to 4.9%.

  Answer:

  – For more complicated bonds (or portfolios) like this one, you need to use *Excel*. For this bond, there are at least two ways you can do this: the Excel DURATION function; or using a spreadsheet like this:

t	CF	PV(CF)	D(CF)	$PV \times D$
1	50	47.6	1	47.6
2	50	45.4	2	90.7
3	50	43.2	3	129.6
4	50	41.1	4	164.5
5	50	39.2	5	195.9
6	50	37.3	6	223.9
7	50	35.5	7	248.7
8	50	33.8	8	270.7
9	50	32.2	9	290.1
10	50	30.7	10	307.0
11	50	29.2	11	321.6
12	50	27.8	12	334.1
13	50	26.5	13	344.7
14	50	25.3	14	353.5
15	50	24.1	15	360.8
16	50	22.9	16	366.5
17	50	21.8	17	370.9
18	50	20.8	18	374.0
19	50	19.8	19	375.9
20	1050	395.7	20	7914.7
Sum:		1000		13085.3

(over)

Calculating the Duration of a Portfolio (4)

Answer (continued):

- The PV's in the 3rd column are just the CF's in the 2nd, divided by $(1+y)^t$, where y is the current yield of 5%.
- The bond price is the sum of the present values of the 20 cashflows (in the 3rd column), which equals \$1000.
- D(CF) is the duration of the individual cashflows, which is shown in the 3rd column. These are just equal to the time until we receive them (*i.e.,* they are equal to t).
- The 4th column ($PV \times D$) is the product of the 2nd and 3rd columns.
- The sum of the values in the 4th column, divided by the bond value, gives the weighted average of the durations of the cashflow-durations. This is the overall duration of the bond, *i.e.,*

$$D = \frac{\sum PV(CF) \times D(CF)}{\text{Portfolio Value}} = \frac{13085.3}{1000} = \boxed{13.085}$$

- This bond's modified duration is:

$$D^* = \frac{D}{1+y} = \frac{13.085}{1.05} = \boxed{12.46}$$

- In response to a change in the yield from 5 to 4.9%, the value of this bond would change by approximately:

$$-12.46 \times -0.1\% = \boxed{1.246\ \%}$$

Calculating the Duration of a Portfolio (5)

- **Example** (A little more complicated): It is 1970. Suppose that you have just started a new Savings & Loan. You have raised \$20 million (in cash) from the firm's equity holders. You have also taken in \$100 million in cash in the form of deposits, and you have made mortgage loans totaling \$80 million in value. All of these mortgage loans are 30 year fixed, and for the sake of simplicity we will assume that the mortgage holders make annual payments, that there is no risk of default, and that the mortgages cannot be prepaid. Also, assume that the yield curve is flat at 4%/year.

  1. Layout the assets and liabilities of the firm.

  2. Calculate the duration of each of the assets and liabilities (excluding Shareholder's Equity).

  3. Calculate the duration of the Shareholder's equity.

  4. Now suppose that the yields at all maturities now increase to 5%. What happens to the value of Shareholders' equity?

  Answer: (See if you can work this out – I will supply the answer later.)

Immunization

- **Definition:** Immunization is the term for adding assets or liabilities to your portfolio so as to make the duration of the portfolio equal to zero, *i.e.*, to *immunize* the portfolio to interest rate risk.
- Other terms for **immunization** are **hedging** and **duration matching**.
- **Example** (continued from page 7): Now suppose that you wish to hedge the interest rate risk of the S&L you are running by buying or selling 30 year zero coupon bonds. Again, assume the yield curve is flat at 4%/year.
  - Suppose that you will buy or sell $100 face value zero coupon bonds. How many should you buy or sell to completely immunize your portfolio?
  - What will your balance sheet (assets, liabilities, and shareholders' equity) look like after you immunize your portfolio?
  - Calculate the duration of the Shareholder's equity, after immunizing the portfolio.

  Answer: (See if you can work this out – I will supply the answer later.)

Duration Facts

There are 6 facts about duration that you need to know from the lecture. The intuition for these is given in the lecture notes. Also, you can understand these via the physical analogy (weights on a board) discussed in class:

The Facts:

1. The duration of a coupon bond is always less than its maturity.

2. For a given coupon rate and yield, a longer maturity (typically) increases the duration of a bond.

3. For a given coupon rate and maturity, a lower yield increases the duration of a bond.

4. For a given yield and maturity, a lower coupon increases the duration of a bond.

5. The duration of a coupon bond is lower when the bond's yield is higher.

6. The duration of a perpetuity is $\frac{1+y}{y}$.

Graphical Approach & Convexity

- The plot on top of page 633 of B&M (Figure 23.4) shows how bond prices change with the interest rate (*i.e.*, with the yield).
  - Notice that the 30 year bond price changes more rapidly than the price of the 1 year.
- The slope of the lines, at an interest rate of 5%, for each of these bonds, is equal to $-D^*$ (their Modified Duration times -1).
  - These bonds all have positive duration, but the slopes of the lines are negative, so we have to multiply the slopes by -1 to get the duration.
- However, this plot also reveals a problem with the duration analysis:
- These lines are not straight – they are curved. Technically, we say that they are **convex**.
  - The reason for this is that, as we know, the duration of these coupon bonds (*i.e.*, the slope of the line) decreases as the yield increases.
  - Mathematically, this means that while our duration equation works pretty well for small changes in the yield, it won't work as well for big changes because the duration changes as the yield changes.
- Moreover, correcting for this is tricky because different bonds have different amounts of convexity.
  - For a zero coupon bond, this plot is almost perfectly flat, but for a coupon bond, it can be very convex.

Convexity

- The way that we can correct for this is to directly calculate the bond convexity (C) using this equation:

$$C = \frac{1}{(1+y)^2} \left\{ (1 \times 2) \times \frac{CF_1/(1+y)}{\text{Bond price}} + (2 \times 3) \times \frac{CF_2/(1+y)^2}{\text{Bond price}} + \ldots + (J \times (J+1)) \times \frac{CF_J/(1+y)^J}{\text{Bond price}} \right\}.$$

  where CF_j is the value of the cashflow received in year j.

- Then, we can incorporate the convexity into the price change equation using this equation:

$$\begin{aligned} \text{\% Change in Bond Price} \;\approx\; & -D^* \times \text{Change in Yield} \\ & + C \times \frac{(\text{Change in Yield})^2}{2} \end{aligned}$$

- Note that for small changes in the yield, for example for a yield change of 0.001 (10 basis points or 0.1%), the change in yield squared will be very small (0.000001), so we can pretty much ignore the convexity.

- However, for big changes it can be important.

- **What do you need to know:** I won't ask you to calculate a bond's convexity, or to calculate a price change that incorporates convexity. However, I might ask you to a question like this:

- **Example:** You hold a 8% coupon bond with 12 years left to maturity. The yield curve is currently flat at 5%. First, based on the duration of this bond, calculate how much, in percent, the price of this bond will decrease if yields on all bonds increase to 6%. Does your answer over- or under-estimate the true change? Explain.

  Answer: The price won't decrease by as much as the standard duration calculation would predict. The reason is the convexity of the bond.