A Note on Constant Proportion Trading Strategies and Leveraged ETFs

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Abstract

Leveraged ETFs (LETFs) have recently been criticized for not performing as 'advertised' and therefore have been deemed as unsuitable for unsophisticated or buy-and-hold investors. In response to this criticism, recent research has shown that the empirical performance of any given LETF is indeed consistent with the stated goal of replicating some multiple of the *daily* performance of the underlying security. In this note we make the obvious connection between LETFs and the classic *constant proportion* trading strategies that have long been studied in the dynamic portfolio optimization literature. While this connection is obvious it does not appear to have been fully utilized in explaining LETF performance. A possible explanation is that the terminal wealth that results from following a constant proportion trading strategy is rarely, if ever, expressed as a function of the terminal values of the underlying assets. Indeed until very recently no such expression has been available in the literature. In this paper we (re)derive such an expression for general diffusion dynamics and multiple underlying securities and use it to explain LEFT performance. In particular we explain why an investor with a long position in an LETF is short realized variance. We also interpret the exposure of a non-leveraged constant-proportion strategy to realized variance as a multiplicative *premium*. This premium compensates an investor for accepting a terminal wealth that is proportional to the generalized geometric mean of the underlying security prices rather than the corresponding arithmetic mean that would be earned by a buy-and-hold investor. We also propose a class of ETFs, *Constant Proportion ETFs* (CPETFs), that should be more suitable for buy-and-hold investors. Moreover, because these CPETFs would sell high and buy low they should help to dampen market volatility at the close, a property not shared by LETFs.

1 Introduction

A *constant proportion* (CP) trading strategy[1] is a strategy in which the fraction of wealth invested in each risky asset is constant and does not vary with time. Such a strategy requires constant rebalancing and is therefore dynamic. CP strategies are perhaps the most well-known of all dynamic trading strategies. They appear as the optimal solution to the classic dynamic portfolio choice problem in which the investment opportunity set does not vary with time and the investor has *constant relative risk aversion*, i.e. a power or logarithmic utility function over terminal wealth and / or intermediate consumption. These problems were first studied and solved by Merton (1969, 1971), Samuelson (1969) and Hakansson (1970). Moreover, the optimality of these strategies is derived in just about every advanced[2] financial economics textbook that discusses dynamic portfolio optimization. Browne (1998) studies the *rate of return on investment* for CP strategies when the underlying securities follow geometric Brownian motions and lists several other problems for which CP strategies are optimal. They include, for example, the problem of minimizing the expected time to reach a given level of wealth and the problem of maximizing the expected discounted reward for reaching a given level of wealth. CP strategies, of course, are also synonymous[3] with the *Kelly Criterion* (Kelly 1956) for optimizing the long-term growth-rate of wealth.

It is remarkable, however, that despite the ubiquity of CP strategies, until very recently we could not find an expression in the literature for the terminal wealth of a CP trading strategy in terms of the terminal security prices. In this note we use the expression[4] for the terminal wealth of a CP strategy that was recently derived by Haugh and Jain (2007) to explain the recent and controversial performance of LETFs. Unlike regular ETFs which are passively managed, LETFs require active management. They have the stated goal of replicating some multiple of the *daily* performance of some underlying security or index. This multiple is greater than one for a positively leveraged ETF and less than zero for an *inverse* ETF. Typical leverage values are ± 2 and ± 3. Many investors who invested in these securities expected returns that would be very similar to the returns of a buy-and-hold investment in the same underlying security at the same leverage multiple. During the highly volatile period of the 2008 credit crisis this was not the case and so LETFs received much attention and criticism from the financial press.

We are certainly not the first to explain LETF performance. Indeed Avellaneda and Zhang (2010) and Cheng and Madhavan (2009) both[5] derived (9) by arguing from first principles. In this paper, we obtain (9) as a particular case of the more general expression derived by Haugh and Jain (2007). We also argue that given an understanding of CP trading strategies, there should have been no surprise whatsoever when LETFs performed as they did during the financial crisis of 2008. Indeed the sensitivity of the performance of a CP trading strategy to realized variance does not appear to be widely known or at the very least, widely appreciated. This is remarkable given the central role they played in the early development of dynamic portfolio optimization and their association with the Kelly Criterion. This lack of appreciation is most likely explained by the fact that just about every treatment of CP strategies in the literature neglects to write the terminal wealth as

[1] These strategies are often referred to as *static* strategies in the literature but we will persist with referring to them as CP strategies in this paper.

[2] For example, see Duffie (2004), Merton (1990) or Cvitanic and Zapatero (2004). See also Karatzas and Shreve (1997) for a more recent treatment of the optimality of constant proportion trading strategies.

[3] See also, for example, Breiman (1961), Thorp (1971), Ethier and Tavare (1983), Ethier (1988), Cover (1991) and Cover and Thomas (1991) for related work.

[4] Haugh and Jain (2007) derived this an expression when security prices have general diffusion dynamics. To the best of their knowledge they were the first to derive this expression despite it being particularly simple to derive.

[5] Cheng and Madhavan (2009) derived (9) assuming geometric Brownian motion dynamics.

a function of the terminal values of the underlying securities. Instead these treatments often end once they have demonstrated the optimality of a CP strategy and on some occasions, derived the optimal value function.

In this note we write the terminal wealth of a CP trading strategy as a function of the terminal security prices and see that it immediately explains LETF performance. Moreover, we can also use this expression to interpret the exposure of a CP strategy to realized variances and covariances as the cost or compensation for following a CP strategy as opposed to a buy-and-hold strategy. We also propose a *constant proportion* ETF (CPETF) and argue that such a security would be more appealing to unsophisticated investors as well as having some positive systemic effects on market microstructure. In particular, the rebalancing requirements of a CPETF would require the manager to sell at the close after an up-day and to buy at the close after a down-day, thereby dampening volatility at the close.

2 Security Price and Wealth Dynamics

We assume there are n risky assets and a single risk-free asset available in the economy. The time t vector of risky asset prices is denoted by $P_t = (P_t^{(1)} \ \dots \ P_t^{(n)})^\top$ and the time t instantaneously risk-free rate of return is denoted by r_t. We assume the price dynamics of the risk assets satisfy

$$\frac{dP_t}{P_t} = \mu_t \, dt + \Sigma_t \, dB_t \tag{1}$$

where dP_t/P_t should be interpreted component-wise, B_t is an m-dimensional standard Brownian motion, μ_t is an n-dimensional adapted process and Σ_t is an $n \times m$ adapted matrix process. It is worth mentioning that our result, i.e. Proposition 1, is no longer valid if we allow jumps in the security prices. Nonetheless, it is straightforward to show that the proposition would would hold approximately[6] in the presence of jumps.

Consider now an investor who chooses to follow a CP trading strategy, $\theta = (\theta_1 \ \dots \ \theta_n)^\top$, so that at any time t, the fraction of wealth invested in the i^{th} risky asset is constant and equal to θ_i. The fraction invested in the cash-account is then given by $1 - \sum_{i=1}^n \theta_i$. The value of the portfolio, W_t, then has the following dynamics

$$\frac{dW_t}{W_t} = \left[(1 - \theta^T \mathbf{1})r + \theta^T \mu_t\right] dt + \theta^T \Sigma_t dB_t. \tag{2}$$

Haugh and Jain (2007) recently found an expression for W_T in terms of P_T when the price dynamics are given by (1). While this expression is trivial to derive (see Proposition 1 below) it has many applications. In particular, we will use it in Section 3 to explain the performance of leveraged ETFs.

[6]Consider for example a jump-diffusion model for the i^{th} security. Then $P_T^{(i)}$ would have an additional factor of the form $\prod_{0<t\leq T}(1+J_t^{(i)})$ where $J_t^{(i)}$ is the relative jump in the price of the i^{th} security at time t. The corresponding term in the expression for the terminal wealth of the CP strategy would be $\prod_{0<t\leq T}(1+\theta_i J_t^{(i)}) \approx \left(\prod_{0<t\leq T}(1+J_t^{(i)})\right)^{\theta_i}$. This later approximation then immediately leads to (3) with the quality of the approximation depending on the actual jump sizes. Given how accurate (3) is in practice (see Figure 2) even in times of extreme market volatility, it is clear we lose very little by working with diffusions.

2.1 The Terminal Wealth of a Constant Proportion Trading Strategy

We have the following proposition[7] which solves for the terminal wealth corresponding to any CP trading strategy when the price dynamics satisfy (1). This result was originally obtained[8] by Haugh and Jain (2007) who used it to study the dual approach for portfolio evaluation that was proposed by Haugh, Kogan and Wang (2006).

Proposition 1 *Suppose price dynamics satisfy (1) and that a static trading strategy is employed so that at each time $t \in [0,T]$ a proportion, θ_i, of time t wealth is invested in the i^{th} risky security for $i = 1,\ldots,n$ with $1-\theta^\top \mathbf{1}$ invested in the risk-free asset. Then the terminal wealth, W_T, resulting from this strategy satisfies*

$$W_T \;=\; W_0\ \exp\left(\int_0^T \left[(1-\theta^\top \mathbf{1})r_t + \frac{1}{2}\theta^\top\left(\mathrm{diag}(\Sigma_t\Sigma_t^\top) - \Sigma_t\Sigma_t^\top\theta\right)\right] dt\right) \prod_{i=1}^n \left(\frac{P_T^{(i)}}{P_0^{(i)}}\right)^{\theta_i} \tag{3}$$

Proof: Using (1) and applying Itô's lemma to $\ln P_T$ we obtain

$$\ln P_T \;=\; \ln P_0 + \int_0^T \left(\mu_t - \frac{1}{2}\,\mathrm{diag}(\Sigma_t\Sigma_t^\top)\right)dt + \int_0^T \Sigma_t\ dB_t. \tag{4}$$

As the wealth dynamics satisfy (2) another simple application of Itô's lemma to $\ln W_T$ then implies

$$W_T \;=\; W_0 \exp\left(\int_0^T \left[(1-\theta^\top\mathbf{1})r_t + \theta^\top\mu_t \;-\; \frac{1}{2}\theta^\top\Sigma_t\Sigma_t^\top\theta\right] dt \;+\; \theta^\top\int_0^T \Sigma_t\ dB_t\right). \tag{5}$$

Substituting (4) into (5) we then obtain (3) as desired. □

Before discussing the relevance of Proposition 1 to LETFs in Section 3, we will briefly discuss some other applications of the proposition.

Merton's Problem

Consider the classic dynamic portfolio optimization problem that was originally considered by Merton (1969). The drift vector, volatility matrix and interest rate are now all assumed to be constant. For an investor with a constant relative risk aversion the optimal solution is to adopt a CP strategy. In this case (3) reduces to

$$W_T \;=\; W_0\ \exp\left((1-\theta^\top\mathbf{1})rT + \frac{1}{2}\theta^\top\left(\mathrm{diag}(\Sigma\Sigma^\top) - \Sigma\Sigma^\top\theta\right)T\right) \prod_{i=1}^n \left(\frac{P_T^{(i)}}{P_0^{(i)}}\right)^{\theta_i} \tag{6}$$

Despite all the attention that has been paid to this problem in the literature, we have only see the expression for W_T in (6) in the recent paper of Haugh and Jain (2007).

Studying Return Predictability

The CP strategy is often used as a base case when studying the value of *predictability* in security prices. Predictability is often[9] induced by setting $\Sigma_t = \Sigma$, a constant, and allowing the drift

[7] This result is so simple to derive that it hardly deserves "proposition" status. Nonetheless, in the absence of any other propositions in this note we will persist with it.

[8] To be precise, Haugh and Jain (2007) assumed that the volatility matrix in (1) was constant but it was clear that their derivation also worked for a stochastic volatility matrix.

[9] See, for example, Lynch (2001).

term, μ_t, to be a function of some state vector process, X_t say. In that case the expression in (6) still applies and terminal wealth is a function of *only* the terminal security prices. Moreover, the expected utility of any CP strategy can often be determined in closed form when the distribution of P_T is also known. For example, if $log(P_t)$ is a (vector) Gaussian process, then W_T is log-normally distributed and

$$V_t := \mathrm{E}_t[W_T^{1-\gamma}/(1-\gamma)] \tag{7}$$

can be computed analytically. Using (7) it is also possible to compute the optimal CP strategy and compare[10] it to the CP strategy that an investor would employ if he ignored the predictability of returns and assumed a constant investment opportunity set.

Equations (6) and (7) can also be used to determine the *myopic* strategy where at each time t, the investor solves his portfolio optimization problem by assuming that the instantaneous moments of asset returns are fixed at their current values for the remainder of the investment horizon. The myopic strategy ignores the hedging component of the optimal trading strategy and has also been studied extensively[11] in the literature.

The Dual Approach to Portfolio Evaluation

Haugh and Jain (2007) used the preceding observations to compute duality-based upper bounds on the value function of the optimal dynamic trading strategy when return predictability was induced via the drift process, μ_t. In addition to improving the efficiency of their numerical algorithms, the closed form expression for the value function in (7) allowed them to construct (in the case of CP strategies) upper bounds on the optimal value function that were superior and theoretically more satisfying than those calculated originally by Haugh, Kogan and Wang (2006).

It is clear then that Proposition 1, while simple, has many potential applications. The final application that we will discuss relates to leveraged ETFs and the controversy surrounding their performance during the credit crisis of 2008.

3 Leveraged ETFs

A leveraged ETF has just a single underlying security or index and promises to track θ times the *daily* performance of this underlying index. This performance is usually achieved through the use of total return swaps. As in Avellaneda and Zhang (2010), we can approximate the value of the LETF with the following stochastic differential equation

$$\frac{dL_t}{L_t} \;=\; \theta\frac{dP_t}{P_t} \;+\; (1-\theta)rdt - fdt \tag{8}$$

where L_t is the time t value of the LETF and f is the constant *expense ratio* of the LETF. The $(1-\theta)rdt$ term in (8) reflects the cost of funding the leveraged position (when $\theta > 1$) or the risk-free

[10]The static strategy obtained from maximizing (7) is the CP strategy that an investor would employ if he was *forced* to select a CP strategy and knew the true price dynamics. In contrast, an investor who was ignorant of the true price dynamics and believed the investment opportunity set was not time-varying would *willingly* select a CP strategy. However, the two CP strategies would not coincide.

[11]See, for example, Kroner and Sultan (1993), Lioui and Poncet (2000), Brooks et al. (2002) and Basak and Chabakauri (2008).

income from an inverse ETF (when $\theta < 0$). Avellaneda and Zhang (2010) and (in the case where P_t follows a geometric Brownian motion) Cheng and Madhavan (2009) solved (8) to obtain

$$\frac{L_T}{L_0} = \left(\frac{P_T}{P_0}\right)^{\theta} \exp\left((1-\theta)rT - fT + \frac{1}{2}\theta(1-\theta)\int_0^T \sigma_t^2 dt\right) \tag{9}$$

and used this expression to explain the empirical performance of LETFs.

The principal motivation for this note is to argue that this performance should have been anticipated in the market given the ubiquity of CP trading strategies in the literature. Indeed if we ignore the expense ratio then it is clear from (8) that the dynamics of L_t are simply those of a CP trading strategy and indeed (3) reduces to (9) in the single risky asset case where we now write σ_t for Σ_t. The expense ratio, being deterministic, simply results in the time T value of the LETF being reduced by a factor of $\exp(-fT)$.

It is worthwhile contrasting (9) with the time T value of a *static* position of θ times the underlying index that was initiated at time $t = 0$. If we denote the time T value of such a position by S_T, then it is clear that

$$\frac{S_T}{S_0} = \frac{\theta P_T - (\theta - 1)\exp(rT)P_0}{P_0}. \tag{10}$$

Many of the original investors in LETFs believed that their returns would resemble the returns in (10) once they had adjusted for the expense ratio, f. And while they would have been justified in this belief in times of low volatility and short investment horizons, the difference between (9) and (10) can be quite remarkable when realized volatility is high.

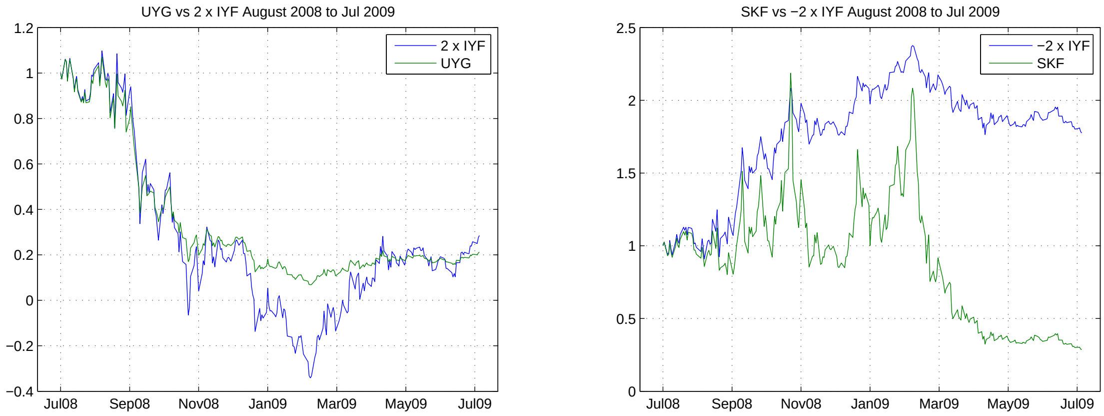
Consider, for example Figure 1(a) where we have plotted the performance of the ProShares Ultra Financial ETF (ticker *UYG*) against a static position of 2 $\times$ the I-Shares Financial Sector ETF (ticker *IYF*) between August 2008[12] and August 2009. The ProShares ETF is a leveraged ETF that is designed to track two times the *daily* performance of the Dow Jones Financial Index (DJFI) while the I-Shares ETF is designed to simply track the DJFI. The discrepancy between the two performances is dramatic and is explained by the very high level of realized variance in that period. Note that an investor in a leveraged ETF with $\theta = 2$ is short realized variance as suggested by (9).

Similarly in Figure 1(b) we have plotted the performance of the ProShares Ultra Short Financial ETF (ticker *SKF*) against a static position of -2 $\times$ the I-Shares Financial Sector ETF, again between August 2008 and August 2009. Note that the ProShares Ultra Short Financial ETF is a leveraged ETF that is designed to track *minus* two times the *daily* performance of the DJFI. The discrepancy between the two is again dramatic and is of course explained by the very high level of realized variance in that period. Note that an investor in a leveraged ETF with $\theta = -2$ is once again short realized variance.

In fact any value of $\theta < 0$ or $\theta > 1$ results in a negative exposure to realized variance for a fixed value of the terminal price of the underlying index, P_T. Moreover the effect is asymmetric in that an LETF with a leverage of $\theta > 1$ is not as short variance as an inverse[13] ETF with a leverage of $-\theta$. This short position in realized variance is best explained by noting that for values of $\theta \notin [0, 1]$ the act of daily rebalancing will require the manager of the LETF to "sell low" and "buy high". The greater the realized variance, the greater the magnitude of rebalancing and so the greater the losses on the LETF. Thus the daily rebalancing is similar to delta-hedging a short position in a vanilla European option where one is also short realized variance. In contrast, an ETF corresponding to a

[12]We assume both positions were entered into at the end of July 2008.

[13]We include inverse ETFs as a subset of LETFs in this note.



(a) ProShares Ultra Financial

(b) ProShares Ultra Short Financial

Figure 1: Performance of LETFs Versus Leveraged Buy-and-Hold Positions in Underlying Index

value of $\theta \in (0, 1)$ is long realized variance and benefits from high levels of realized variance, again conditional on P_T. We will return to this issue again in Section 5.

Our discussion of the performance of leveraged ETFs has implicitly assumed that their returns are well approximated by (9). As demonstrated by Avellaneda and Zhang (2010) this is indeed the case, even when markets are highly volatile. They analyze the tracking error when the actual performance of LETFs is approximated by (9) and conclude that even in very volatile markets, the error is small. For example, in Figure 2(a) we graph the performance of the ProShares Ultra Financial LETF ($\theta = 2$) against the performance implied by (9). We assumed[14] in the latter case that $r = f = 1\%$. Note that that the two graphs are in extremely close agreement with one another despite the very high levels of realized volatility during that period.

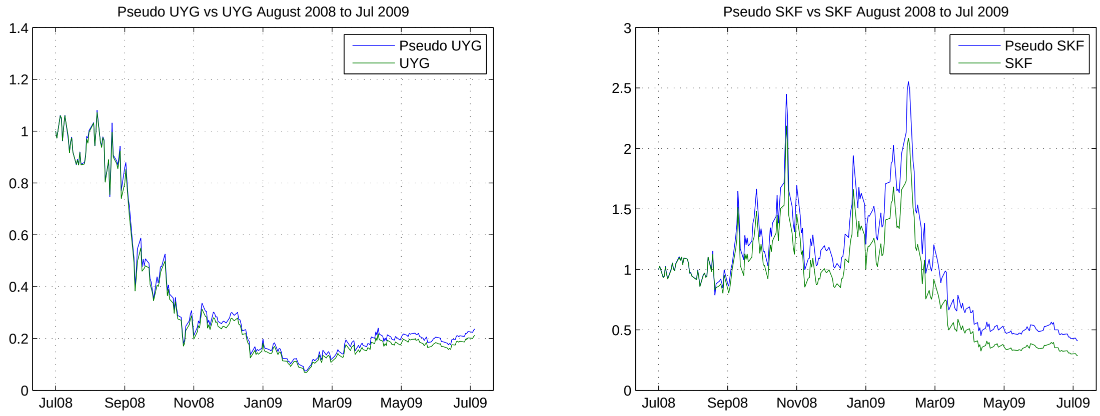
Similarly in Figure 2(b) we graph the performance of the ProShares Ultra Short Financial LETF ($\theta = -2$) against the performance implied by (9). We again assumed in the latter case that $r = f = 1\%$. While the two graphs are very similar there is nonetheless a clear discrepancy between the two which Avellaneda and Zhang put down to the difficulty in shorting financial stocks during this period. Of the 56 LETFs considered by Avellaneda and Zhang, this was atypical and the tracking error was generally closer to that of Figure 2(a).

Before concluding this section, it is worth mentioning the effects[15] that the presence of LETFs can have on market microstructure. Because LETFs need to buy at the close when the market is up and sell at the close when the market is down, they have been blamed[16] for increasing volatility at the close. Furthermore, because the direction of the daily rebalancing trades are widely known in the market, it is suspected that many proprietary trading desks have regularly front-run these trades. They are therefore suspected of adding to market volatility at the close as well as negatively impacting LETF performance. Cheng and Madhavan (2009) provide an account of these microstructure effects and estimate the aggregate daily hedging demand of LETFs in the market.

[14]Over this period the 1-month risk free rate moved from approximately 1.6% to .1% but we find using a constant rate of 1% makes no discernable difference to the results.

[15]Variance-swaps are somewhat similar in that their hedging also requires daily rebalancing at the close.

[16]See, for example, Lauricella, Pulliam and Gullapalli (2008).



(a) ProShares Ultra Financial

(b) ProShares Ultra Short Financial

Figure 2: Actual Performance of Leveraged ETFs Versus Performance Predicted by (9)

4 The CP Strategy Versus the Buy-and-Hold Portfolio

We will now refer to the risk-free asset as the 0^{th} security and use θ_0 to denote the fraction of wealth invested in this security. Unless otherwise stated we will assume that $0 \le \theta_i \le 1$ for each $i = 0, \ldots, n$ and that $\sum_{i=0}^{n} \theta_i = 1$. Consider now a buy-and-hold strategy where at time $t = 0$ we invest a constant proportion, θ_i, of our time $t = 0$ wealth in the i^{th} security for $i = 0, \ldots, n$. Assuming we started with an initial wealth of W_0, then the gross return at date T is given by

$$\frac{W_T}{W_0} = \sum_{i=0}^{n} \theta_i \frac{P_T^{(i)}}{P_0^{(i)}}. \tag{11}$$

Similarly we can rewrite[17] (3) as

$$\begin{aligned} \frac{W_T}{W_0} &= \exp\left(\frac{1}{2}\theta^\top \int_0^T \left(\mathrm{diag}(\Sigma_t^a \Sigma_t^{a\top}) - \Sigma_t^a \Sigma_t^{a\top} \theta\right) dt\right) \prod_{i=0}^{n} \left(\frac{P_T^{(i)}}{P_0^{(i)}}\right)^{\theta_i} && (12) \\ &= \exp\left(\frac{1}{2}\int_0^T \left(\sum_{i=0}^{n} \theta_i \sigma_{t,i}^2 - \mathrm{Var}(\theta^\top R_t)\right) dt\right) \prod_{i=0}^{n} \left(\frac{P_T^{(i)}}{P_0^{(i)}}\right)^{\theta_i} && (13) \end{aligned}$$

where Σ_t^a is the instantaneous variance-covariance matrix of the $n+1$ securities, $R_t = (R_t^{(0)}, \ldots, R_t^{(n)})$ is the time t vector of their instantaneous[18] returns and $\sigma_{t,i}^2 := \mathrm{Var}(R_t^{(i)})$. It is particularly interesting now to compare (11) and (13). In fact we can apply the general arithmetic-geometric mean inequality to conclude that

$$\sum_{i=0}^{n} \theta_i \frac{P_T^{(i)}}{P_0^{(i)}} \ge \prod_{i=0}^{n} \left(\frac{P_T^{(i)}}{P_0^{(i)}}\right)^{\theta_i}. \tag{14}$$

[17]Note that we now use θ to denote $(\theta_0, \ldots, \theta_n)^\top$.

[18]That is, $R_t^{(i)} = dP_t^{(i)}/P_t^{(i)}$. Note also that the first row and first column of Σ_t^a contains only zeros and the sub-matrix beginning at the $(2,2)^{th}$ element is identical to Σ_t.

It is also straightforward to show that

$$\sum_{i=0}^{n} \theta_i \sigma_{t,i}^2 \; - \; \mathrm{Var}(\theta^\top R_t) \;\; \geq \;\; \sum_{i=0}^{n} \theta_i \sigma_{t,i}^2 \; - \; \left(\sum_{i=0}^{n} \theta_i \sigma_{t,i} \right)^2 \; \geq \; 0 \tag{15}$$

for all t and so it follows that the exponential term in (13) is always greater than or equal to 1.

Compensation for Earning the Geometric Mean

We can therefore interpret the exponential term in (13) as the (multiplicative) compensation that an investor receives for accepting the geometric mean of a CP strategy instead of the arithmetic mean of the corresponding buy-and-hold strategy. This compensation is similar to holding a regular option in that the CP strategy is long *gamma*: it therefore profits from the act of rebalancing by selling high and buying low.

Moreover an investor in a CP trading strategy benefits from knowing that the geometric mean of the underlying security returns will constitute a lower bound on his overall portfolio return. The degree to which the realized return outperforms this lower bound will depend on the realized variances and covariances of the securities. This long volatility feature of CP strategies has been (at least informally) identified by others. For example, Luenberger (1997) demonstrates how an investor can benefit from volatility by rebalancing his portfolio in each period and he refers to this phenomenon as *volatility pumping*. More generally, the large literature on the Kelly Criterion[19] and proportional betting has long been aware of this fact. But as stated earlier, the expression in (12) does not seem to be widely known and though it is simple to derive, the link between the geometric and arithmetic means also seems to be new.

While these observations apply whenever $0 \leq \theta_i \leq 1$ and $\sum_{i=0}^{n} \theta_i = 1$, they can also apply for certain θ vectors that do not satisfy these constraints. For example, suppose $\Sigma_t = \Sigma$ is a constant matrix and that we maximize the exponential term in (13) over $(\theta_0, \ldots, \theta_n)$. Setting $\sigma_i^2 := \mathrm{Var}(R_t^{(i)})$, it is again straightforward to see that

$$\theta^* \; = \; \frac{1}{2} \left(\Sigma \Sigma^\top \right)^{-1} [\sigma_1^2 \; \cdots \; \sigma_n^2]^\top$$

is the maximizing vector corresponding to the n risky securities with $1 - \sum_{i=1}^{n} \theta_i^*$ denoting the proportion invested in the risk-free security. With this strategy it is easy to check that the exponential term in (13) reduces to

$$\exp\left(\frac{T}{4} \, [\sigma_1^2 \; \cdots \; \sigma_n^2] \left(\Sigma \Sigma^\top \right)^{-1} [\sigma_1^2 \; \cdots \; \sigma_n^2]^\top \right) \; \geq \; 1. \tag{16}$$

The left-hand-side of (16) is greater than or equal to 1 as the inverse of a positive-definite covariance matrix is also positive-definite. Note however, that there is no reason why some components of θ^* cannot be negative or exceed 1. As a result, it is possible that the CP strategy with the greatest, i.e. most positive, exposure to realized variances and covariances requires short selling and leveraged positions in the underlying securities. This is perhaps surprising given the results on LETFs when there is only one risky security, i.e. when $n = 1$. In this $n = 1$ case the general arithmetic-geometric mean inequality no longer applies and now the expression in (13) is the (multiplicative) premium that you must pay for following the CP strategy, i.e. for purchasing the LETF.

[19] See for example the references listed in Section 1.

5 A Constant Proportion ETF?

Given the inability of most investors to time the market, constant proportion trading strategies should, at least in the absence of market frictions, be reasonably close to optimal for investors with power or logarithmic utility. The costs[20] associated with daily rebalancing, however, would be prohibitively expensive and time-consuming for individual investors. It might be possible, however, for an actively managed ETF to employ such a strategy. It would be similar to a regular LETF only instead of one underlying risky security, there could be n underlying risky securities. For example, an investor wishing to invest in global equity markets might be interested in an ETF that tracks the daily returns of the S&P 500, the Eurostoxx 50 and the Nikkei 225. In this case we would have $n = 3$. Moreover, if $0 \leq \theta_i$ for $i = 1, 2, 3$ and $\sum_{i=1}^{3} \theta_i \leq 1$, then we know such a product would have a long exposure to market volatility, in contrast to LETFs. Such a product, a *constant proportion* ETF (CPETF) say, could be suitable for unsophisticated investors.

In addition, the manager of a CPETF would necessarily sell at the close after an up-day and buy at the close after a down-day and would therefore tend to dampen market volatility at the close. If rebalancing costs were too expensive, then the CPETF cold be allowed to rebalance less frequently, say once a week or once a month. Or alternatively, it might be required to be balanced at the close only one day a month. This would make it difficult for proprietary trading desks to front-run the CPETF manager. Of course, such a CPETF would only be permitted as long as it satisfied the transparency requirements imposed by various regulatory agencies. While less frequent rebalancings would render (12) a less useful approximation, the insights from (12) should still apply.

6 Conclusions

In this note we utilized the connection between LETFs and constant proportion trading strategies to explain the recent performance of leveraged ETFs . We did this by (re)deriving an expression for the terminal wealth of a CP strategy in terms of the terminal values of the underlying securities under general diffusion dynamics. We explained why an investor with a long position in an LETF is short realized variance and we also interpreted the exposure of a non-leveraged CP strategy to realized variance as a multiplicative premium. This premium compensates an investor for accepting a terminal wealth that is proportional to the generalized geometric mean of the underlying security prices rather than the corresponding arithmetic mean that would be earned by a buy-and-hold investor. We also proposed the class of constant proportion ETFs, a class that should be more suitable for buy-and-hold investors. Moreover, because these ETFs would sell high and buy low they should help to dampen market volatility at the close, a property not shared by LETFs.

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[20]For most investors such strategies would also be inefficient from a tax perspective.

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