Variance Term Structure and VIX Futures Pricing[1]

Yingzi Zhu[2]
Department of International Trade and Finance
School of Economics and Management
Tsinghua University
Beijing 100084, China
Email: zhuyz@em.tsinghua.edu.cn

Jin E. Zhang
School of Business
and School of Economics and Finance
The University of Hong Kong
Pokfulam Road, Hong Kong
Email: jinzhang@hku.hk

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Abstract

Using no arbitrage principle, we derive a relationship between the drift term of risk-neutral dynamics for instantaneous variance and the term structure of forward variance curve. We show that the forward variance curve can be derived from options market. Based on the variance term structure, we derive a no arbitrage pricing model for VIX futures pricing. The model is the first no arbitrage model combining options market and VIX futures market. The model can be easily generalized to price other volatility derivatives.

Keywords: Stochastic volatility; Variance term structure; Arbitrage-free model; Volatility derivatives; VIX futures.

JEL Classification Code: G13

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[2]Corresponding author. Tel: (86)10-6278-6041, Fax: (86)10-6278-4554. Yingzi Zhu thanks Tsinghua University for financial support under Seed Funding for research. Jin E. Zhang thanks the University of Hong Kong for financial support under Seed Funding for Basic Research.

1 Introduction

It has been well documented that both the equity returns and variances are random over time, and they are negatively correlated, see e.g., French, Schwert and Stambaugh (1987). Portfolio managers with long positions on equity are concerned that volatility will increase, which is correlated with negative equity returns. They would seek an asset that has positive payoffs when volatility increases in order to hedge against this risk. When investing in a volatility[3] sensitive security such as stock index options or options portfolios, an investor faces not only return variance risk, but also the (leveraged) stock price risk. To trade views on volatility, or to manage variance risk, it is important for investors to trade volatility directly.

Roughly speaking, there are two ways to trade views on volatility or manage volatility risk. One way to trade volatility is to buy ATM options or straddles. But options or straddles do not always stay at-the-money. Out-of-money or in-the-money options has smaller Vega or volatility sensitivity, which, as observed in Zhu and Avellaneda (1998), will not satisfy investor's need for volatility risk management because there may not be enough volatility to buy when market goes down. In addition, options bundle volatility risk together with price risk, which makes it inefficient and inconvenient to manage volatility risk.

Another way to trade volatility is to use the over-the-counter variance swap market. The corresponding volatility of a variance swap rate is usually called variance swap volatility (VSV). As observed in Derman (1998), variance swap can be priced without making any assumption on the evolution of the volatility process. In fact, the variance swap can be statically replicated by a portfolio of options, plus a dynamic hedging position in underlying futures. The value of VSV is directly linked to the value of a portfolio of options. Due

[3]For convenience, in this article, we use variance and volatility interchangeably, with an understanding that volatility is the square root of the corresponding annualized variance.

to its model independent nature, and its clear economic meaning, the VSV has become a benchmark for analyzing options in general and volatility skew in particular.

On September 22, 2003, CBOE started to publish the 30 day VSV on S&P 500 index (SPX) options, under symbol VIX, and back-calculated the VIX up to 1990 based on historical option prices. The detailed calculation formula is based on the value of an option portfolio.[4] On March 26, 2004, the CBOE launched a new exchange, the CBOE Futures Exchange (CFE) to start trading futures on VIX. The CBOE is now developing a volatility derivative market by using the VIX as the underlying.

Most of the current literature on volatility derivatives focus on the pricing under risk-neutral probability of variance, taking a stochastic volatility model as starting point, see Howison, Rafailidis, and Rasmussen (2004) and the references therein. This approach disconnects the options market and volatility derivatives market. In particular, the correlation between variance and the price process does not enter the pricing formula. On the other hand, our model is based on arbitrage argument between options market and pure volatility derivatives market. The contribution of this paper is to derive an arbitrage-free pricing model based on the corresponding options market. In other words, the model precludes arbitrage opportunity between options market and pure volatility and its derivatives market. The assumption of the theory is the effective integration between these two markets. The risk premium thus implied from the options market depends on the volatility skew of the market. This is the most important feature in our model. Our model answers the important question of how volatility skew of options market affects the price of volatility derivatives. It has been well documented in empirical literature that the variance risk premium in S&P 500 index is negative due to negative correlation between the index return and implied volatility, e.g. Bakshi, Cao, Chen (1997), and Bakshi and Kapadia (2003). Therefore, the risk neutral stochastic volatility drift term thus implied from index options

[4]Refer to the CBOE VIX white paper.

market should have correlation information coded in.

The theory draw strong similarity from arbitrage-free interest rate term structure models. Due to the simple economic meaning of variance swap rate, one can obtain arbitrage-free variance term structure from the corresponding options market. If the options market is complete, in the sense that there exists one call option on any combination of strike price and time to maturity, then the arbitrage-free variance term structure is unique. We note that even if the options market is complete, one still need additional information on the variance of variance to model other volatility derivatives, for example, the OTC volatility swaps, the exchange traded VIX futures, or potential product such as options on VIX. However, due to the incompleteness of the options market, there are infinitely many variance term structures that can be implied by the options market. Similar to interest rate term structure modelling, one needs an interpolation model or dynamic model to "complete" the market. We propose to use Weighted Monte Carlo method (WMC) to infer a unique forward variance term structure from options market. The method is well documented in Avellaneda, et al (2000). The attractive feature of WMC application in variance term structure model is that it combines historical volatility time series information with the current options market information. As widely experienced in interest rate term structure modelling, the proper combination of the arbitrage-free and equilibrium approaches is an important part of the art of term structure modelling.

Dupire (1993) attempted to develop an HJM type arbitrage volatility model, where it starts from an assumption on forward variance swap rate term structure, and derives arbitrage-free instantaneous volatility dynamics. In our paper, we take another route. We start from a process for instantaneous volatility. With a variance term structure derived from options market, we derive the no arbitrage drift term. We believe that our approach is more practical because instantaneous volatility has been the object of interest for many popular stochastic volatility model so far. In addition, instantaneous volatility time series is more readily available than that of forward variances. Our model is similar to a family of single factor short-term interest rate term structure model, such as Ho-Lee model (1986) and Hull and White (1990).

Although the model is presented in a single factor formulation, it can be easily generalized to multi-factor model. In fact, as the time series study of FX options market documented in Zhu and Avellaneda (1998), the FX volatility term structure can be well approximated by a three factor model. In terms of pricing, similar to a vast literature on interest rate derivatives pricing, the number of factors to be included should be determined by the applications at hand. In a previous research, Zhang and Zhu (2005) has documented the need to include an additional factor to fit the observed VIX futures prices. Our paper shows that, without an additional factor, we can also fit the VIX futures price by including a deterministic time-varying mean reversion level of instantaneous variance.

The rest of the paper is structured as follows. In section 2, we derive the arbitrage-free pricing model for volatility derivatives in general, based on market observed option prices. As an important application, we derive an arbitrage-free pricing model for VIX futures. Section 3 we show how to calibrate variance term structure to options market by using WMC method. Based on this term structure, we are able to price the VIX futures. We make comparison with previous research with popular stochastic volatility model, and show that with this model, we can not only capture the level of variance term structure, but also the shape of the term structure. We draw discussion and conclusion in section 4.

2 Arbitrage Pricing Model for Volatility Derivatives

The basic building block of an arbitrage pricing model for volatility derivatives is the variance swap. Assume the stochastic differential equation followed by the stock or stock index of which the volatility is being modelled as:

$$\frac{dS_t}{S_t} = \mu_t dt + \sqrt{V_t} dB_t^1 \tag{1}$$

where the B_t^1 is a standard Brownian Motion. By Ito's lemma,

$$d \ln S_t = \frac{dS_t}{S_t} - \frac{\sqrt{V_t}}{2} dt, \tag{2}$$

which, integrated between T_1 and T_2, yields

$$\ln S_{T_2} - \ln S_{T_1} = \int_{T_1}^{T_2} \frac{dS_t}{S_t} - \frac{1}{2} \int_{T_1}^{T_2} V_t dt, \tag{3}$$

which we can rewrite as

$$\int_{T_1}^{T_2} V_t dt = 2 \int_{T_1}^{T_2} \frac{dS_t}{S_t} - 2(\ln S_{T_2} - \ln S_{T_1}). \tag{4}$$

The stochastic integral $\int_{T_1}^{T_2} \frac{dS_t}{S_t}$ can be interpreted as a self-financing strategy of the underlying stock, and the payoff of $\ln S_T$ is the so-called log contract. As first observed by Breeden and Litzenberger (1978), the log contract can be exactly replicated by a continuum of European options, which can be approximated by a discrete set of European options. CBOE chose to use only market traded options as discrete approximation of exact replication of log contract. In this sense, CBOE's methodology is the same as log contract replication. In this paper, we use WMC to generate option prices to replicate log contract on a continuum of expiration dates and strike prices.

2.1 Term Structure of Instantaneous Variance

From above discussion, with the price of log contract denoted as $L_T(t)$, i.e.,

$$L_T(t) = E_t^Q (\ln S_T),$$

where Q is the risk-neutral probability measure, we can define the *forward variance* from T_1 to T_2 observed at time t as

$$V_{T_1}^{T_2}(t) = \frac{1}{T_2 - T_1} E_t^Q \left(\int_{T_1}^{T_2} V_t dt \right) = 2(r - q) - 2 \frac{L_{T_2}(t) - L_{T_1}(t)}{T_2 - T_1}, \tag{5}$$

where r and q are interest rate and dividend yield. When taking the limit $T_2 \to T_1 = T$, we get the *instantaneous forward variance* observed at time t defined as follows.

Definition 1. The *instantaneous forward variance* $V_T(t)$ as observed at time t is defined as

$$V_T(t) = \lim_{T_2 \to T_1 = T} V_{T_1}^{T_2}(t) = E_t^Q \left(\lim_{\Delta T \to 0} \frac{1}{\Delta T} \int_T^{T+\Delta T} V_s ds \right) = E_t^Q(V_T). \tag{6}$$

Based on the above definition, the instantaneous variance $V_t = \lim_{T \to t} V_T(t)$. Note that the instantaneous forward variance is similar to the instantaneous forward rate in term structure literature, while the instantaneous variance is similar to the instantaneous short term interest rate. Based on the above definition, we proceed with the arbitrage-free model of volatility derivatives.

2.2 The One-factor Arbitrage-free Pricing Model

Given the instantaneous variance term structure $V_T(0)$ observed at time $t = 0$, assume some smoothness condition, we have the relationship between instantaneous variance curve and the mean-reversion level of instantaneous variance.

Proposition 1. If the risk-neutral instantaneous volatility follows a square root process, i.e.,

$$dV_t = \kappa(\theta(t) - V_t)dt + \sigma\sqrt{V_t}dW_t, \tag{7}$$

then the no arbitrage condition requires that

$$\theta(T) = V_T(t) + \frac{dV_T(t)}{\kappa dT}, \tag{8}$$

where $V_T(t)$ is the instantaneous forward variance term structure at time t. Or

$$V_T(t) = V_t e^{-\kappa(T-t)} + \kappa \int_t^T e^{-\kappa(T-s)} \theta(s) ds. \tag{9}$$

The implication of the proposition is that, if variance is stochastic and follows a Heston model, the forward variance can be calibrated (assuming a time-dependent, non-stochastic risk premium) by modifying the drift of the volatility process. The risk premium thus implied from the options market depends on the volatility skew of the market. This is the most important feature of the arbitrage model. Currently, most of the other literature on volatility derivative pricing (Howison, Rafailidis, and Rasmussen, 2004) starts from risk-neutral volatility process. This approach disconnects the options market and volatility derivatives market. In particular, the correlation between variance and the price process does not enter the pricing formula. On the other hand, our model is based on arbitrage argument between options market and pure volatility derivatives market, which is well positioned to answer the important question of how volatility skew of options market affects the price of volatility derivatives. It has been well documented in empirical literature that the variance risk premium in S&P 500 index is negative due to negative correlation between the index return and implied volatility, e.g. Bakshi, Cao, Chen (1997), and Bakshi and Kapadia (2003). Therefore, the risk neutral stochastic volatility drift term thus implied from index options market should have correlation information coded in.

Before proceeding, let's state two properties of the risk neutral drift of the stochastic variance.

Corollary 1. If the instantaneous forward variance term structure has the form of $V_T(0) = \theta_0 + \theta_1 e^{-\kappa T}$ for some constants θ_0 and θ_1, we obtain flat mean reversion level $\theta(t) = \theta_0$. In particular, if the instantaneous forward variance $V_T(0) = V_0$ is flat, We have

$\theta(T) = V_0$.

Corollary 2. When the speed of mean reversion is large, i.e., $\kappa \gg 1$, then $\theta(T) \approx V_T(0)$.

By option prices only, we can only retain the relationship between κ and $\theta(t)$. In fact, there is a more general relationship if we drop the assumption of constant mean reversion rate κ. In a more general case, we have $\kappa(t)\theta(t) = \kappa(t)V_t(0) + \frac{dV_t(0)}{dt}$. In order to obtain κ and θ, one need to have a specific form of risk premium, as well as historical time series model for instantaneous variance V_t. For purpose of parameter estimation, we make the following assumptions on the physical process for V_t and the variance risk premium:

1. The physical process of V_t follows Heston's model:

$$dV_t = \kappa(\theta_0 - V_t)dt + \sigma_V \sqrt{V_t} dW_t \tag{10}$$

2. The risk premium is postulated as a function of time only, namely,

$$\lambda(t)\sigma_V \sqrt{V_t} = \kappa(\theta_0 - \theta(t)) \tag{11}$$

Note that this is in contrast to standard specification for risk premium as $\lambda\sqrt{V_t}$, e.g., Heston (1993). With this specification, the mean reversion speed parameter κ can be estimated from the physical process. We use maximum likelihood estimation for parameter estimations. Interested readers should refer to Appendix for details.

With the calibration of instantaneous forward variance term structure, we develop an arbitrage-free model for VIX futures. Any arbitrage-free model has to observe the current market prices. In VIX futures pricing, one needs to price the current options market correctly. In our setting, we require the model to be able to price the current forward variance curve correctly.

2.3 VIX Futures Pricing

Under risk-neutral probability measure, the SPX and variance dynamics can be written as:

$$\frac{dS_t}{S_t} = (r-q)dt + \sqrt{V_t}dB_1(t), \tag{12}$$

$$dV_t = \kappa(\theta(t) - V_t)dt + \sigma_V\sqrt{V_t}dB_2(t), \tag{13}$$

where $\theta(t)$ is obtained by options market.

The relation between VIX_t^2 and V_t can be derived from the definition of VIX,

$$VIX_t^2 = E_t^Q\left[\frac{1}{\tau_0}\int_t^{t+\tau_0} V_s ds\right], \tag{14}$$

where τ_0 is 30 calendar days. We have the following result for VIX squared.

Proposition 2. With instantaneous variance V_t given by (7), the VIX squared value at time t is given by

$$VIX_t^2 = A + BV_t, \tag{15}$$

where

$$A = \frac{1}{\tau_0}\int_0^{\tau_0}(1 - e^{-\kappa(\tau_0-\tau)})\theta(t+\tau)d\tau, \tag{16}$$

$$B = \frac{1-e^{-\kappa\tau_0}}{\kappa\tau_0}, \tag{17}$$

and $\tau_0 = 30/365$.

To price VIX futures, we need to find the conditional probability density function $f^Q(V_T|V_0)$. With the instantaneous variance process following the SDE given by equation (13), the corresponding risk-neutral probability density $f^Q(V_T|V_t)$ can be determined. Since

$$E_t^Q(e^{uV_T}) = e^{\alpha(t,u)+\beta(t,u)V_t}, \tag{18}$$

where $\alpha(t,u)$ and $\beta(t,u)$ are given by:

$$\beta(t,u) = \frac{\kappa u e^{-\kappa(T-t)}}{\kappa - \frac{1}{2}\sigma^2 u(1-e^{-\kappa(T-t)})}, \tag{19}$$

$$\alpha(t,u) = \kappa \int_t^T \theta(s)\beta(s,u)ds \tag{20}$$

The characteristic function of the risk-neutral instantaneous variance is

$$E_t^Q(e^{i\phi V_T}) = e^{\alpha(t,i\phi)+\beta(t,i\phi)V_t}. \tag{21}$$

Denote $\bar{\theta} = \frac{1}{T-t}\int_t^T \theta(s)ds$, We have the following proposition for the risk-neutral density function $f^Q(V_T|V_t)$.

Proposition 3. With condition

$$\kappa\bar{\theta} > \frac{1}{2}\sigma^2, \tag{22}$$

the risk-neutral probability density function $f^Q(V_T|V_t)$ is well defined by the following invert transformation of its characteristic function given by (21) as follows:

$$f^Q(V_T|V_t) = \frac{1}{\pi}\int_0^\infty Re\left[e^{-i\phi V_T+\alpha(t,i\phi)+\beta(t,i\phi)V_t}\right] d\phi \tag{23}$$

with α and β given by (20) and (19).

With constant $\theta(t)$, we get the standard non-central χ-square distribution (Cox, Ingersoll, and Ross 1985).

Proposition 4. The VIX futures with maturity T is priced as

$$F_T(0) = E_0^Q(VIX_T) = \int_0^{+\infty} \sqrt{A+BV_t} f^Q(V_T|V_0)dV_T, \tag{24}$$

where A and B are given by (16, 17), and $f^Q(V_T|V_t)$ is given by (23).

For proof of the above propositions, we refer interested readers to the Appendix. When $\theta(t)$ becomes constant, we get the case studied in Zhang and Zhu (2005).

In the next section we first calibrate the forward variance curve with options data by WMC method. Using the resulted risk-neutral process we price the VIX futures.

3 The VIX Futures Market Data and Calibration

3.1 Market Data and Calibration Methodology

WMC is a general non-parametric approach developed for calibrating Monte Carlo models to benchmark security prices. It has been used to options market to price volatility skewness, e.g., Avellaneda et al, 2000. WMC starts from a given model for market dynamics, which is usually the empirical probability measure, the prior. Model calibration is done by assigning different weights to the paths generated by the prior probability. The choice of weights is done by minimizing the Kullback-Leibler relative entropy distance of the posterior measure to the prior measure. In this way, we get the risk-neutral measure that is consistent with the given set of benchmark securities. Generally speaking, in an incomplete market, there are an infinite number of such probability measures that fit the market. WMC is a method prescribed to find among the feasible set of probability measures that is "closest" to the prior measure.

As discussed in Avellaneda, et al 2000, the procedure of WMC is as follows:

1. Generate ν paths by Monte Carlo based on the prior measure P.

2. For N benchmark securities, compute cashflow G_j for each of the N securities, with market prices $C_j, j = 1, ..., N$. By optimize

$$Min_{q_1,\ldots,q_\nu} D(q|p) \tag{25}$$

$$s.t. E^Q(G_j) = C_j \tag{26}$$

where $D(q|p) = \sum_{i=1}^{\nu} q_i \ln(\frac{q_i}{p_i})$ is the Kullback-Leibler relative entropy distance from probability measure Q to P.

3. Using the obtained "risk-neutral" probability Q to price other derivative securities.

Specifically, we use Heston model (Stephen Heston, 1993) as the prior. We use a novel Maximum Likelihood estimation method to estimate the parameters. The MLE estimation details are presented in the Appendix. We take full advantage of the historical time series of VIX published by CBOE from 1990 to 2005, and the S&P 500 index level to estimate the instantaneous variance time series.

As example, we use S&P 500 index options price and corresponding VIX futures price on March 10, 2005. To fit the variance term structure, we use OTM options only, because VIX is being calculated using OTM options. In addition, we use options with maturity between 30 days and 1 year, which is the range of maturity the VIX futures are traded. We choose trading volume bigger than 1000 contracts. There are 35 puts and calls chosen to fit the term structure. The options used to calibrate the variance term structure is listed in Table 2.

3.2 Variance Term Structure Calibration using WMC Method

Use WMC, we fit the market prices of options to obtain the forward variance term structure. The fitted instantaneous forward variance term structure is presented in Figure 1. We have converted the variance to volatility for comparison.

The corresponding risk-neutral mean reversion level for the instantaneous variance process, $\theta(t)$ is presented in Figure 2.

3.3 Pricing VIX Futures

Using the calibrated model, we get the price for the VIX futures series on March 10, 2005, as in Table 1. Model price corresponds to the fitted model with time varying mean-reverting level $\theta(t)$. Model1 and Model2 corresponds to constant mean-reverting level of 0.027 and

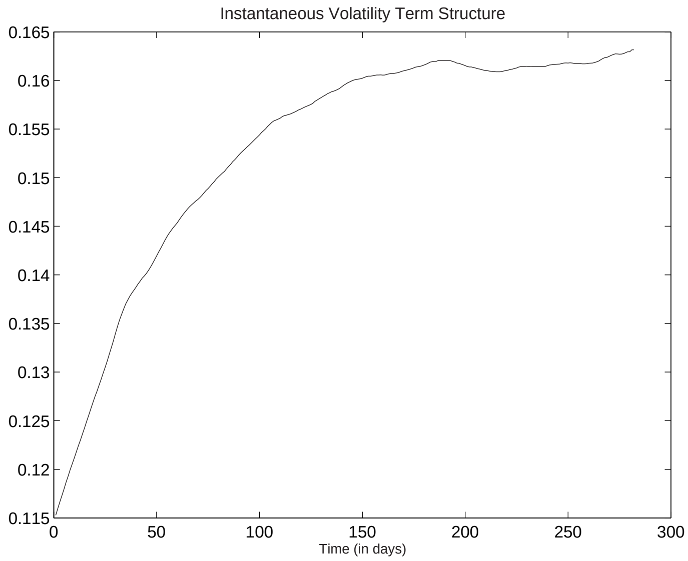

Figure 1: Instantaneous forward Volatility Term Structure fitted from S&P 500 index options market prices on March 10, 2005, by WMC. Note that the empirical long term mean-reversion level of the instantaneous variance is $17\%^2 = 0.0299$. And the VIX level on March 10, 2005 is 12%. The empirical mean-reversion half life is 2 to 3 months.

0.024, respectively. The market and model comparison is also presented in Figure 3. We can see that the varying mean reversion model captures the market prices better than Heston model with constant mean reversion level.

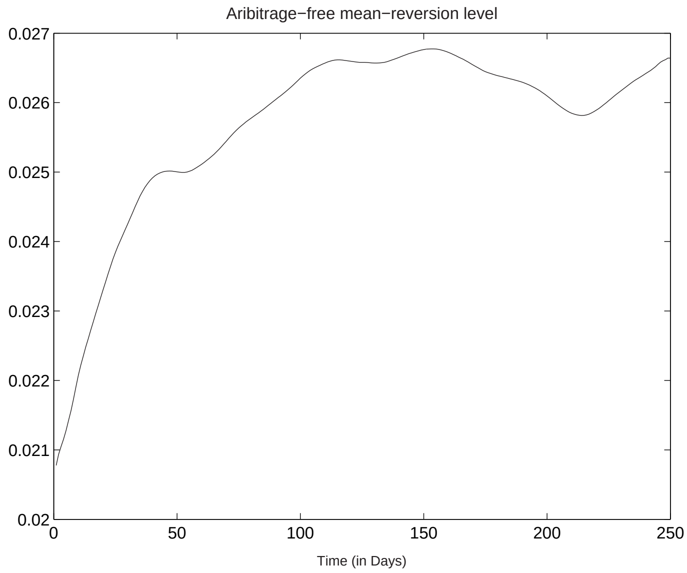

Figure 2: $\theta(t)$ or the risk-neutral mean-reversion level derived from volatility term structure on March 10, 2005. Note that $\theta(t)$ increase from $14.5\%^2 = 0.021$ to $16.4\%^2 = 0.027$. This better fits the market for VIX futures than constant long term mean. The ruggedness is due to the differentiation with respect to term T in equation (8). The first order derivative is taken after linear smoothing.

4 Conclusion

We have developed an arbitrage-free pricing model for volatility derivatives, in particular, we price VIX futures using the derived model. We show that in order to exclude arbitrage opportunity between options market and corresponding volatility derivatives market, the drift term of risk-neutral process of instantaneous variance cannot be determined arbitrarily. In particular, the drift term (or equivalently, the form of risk premium implied therein) can be uniquely determined by the *forward variance curve.* We use WMC method to calibrate the variance term structure for S&P 500 index options market, and priced the VIX futures based on the derived arbitrage-free model. We show that the shape of the variance term structure has major impact on VIX futures pricing. Further research will involve alternative method or improved WMC method to derive variance term structure from options market.

	Maturity (Days)	Market Price	Model Price	Model1	Model2
VIX/H5	6	127.4	126.3	127.3	126.3
VIX/K5	69	135.5	136.1	144.9	138.2
VIX/Q5	160	140.2	144.4	154.0	145.1
VIX/X5	251	151.0	152.5	156.6	147.1
MSE			2.32	8.81	3.45

Table 1: Model Price corresponds to the fitted market. Model price corresponds to the fitted model with time varying mean-reverting level $\theta(t)$. Model1 and Model2 corresponds to constant mean-reverting level of 0.027 and 0.024, respectively. Mean Squared Error is calculated for each model with respect to market price. The pricing error of the constant mean reversion models cannot be reduced due to the rigidity of corresponding variance term structure.

Furthermore, variance term structure of other index options as well as empirical studies on variance term structure will be interesting for the derivatives market on VIX that is being developed.

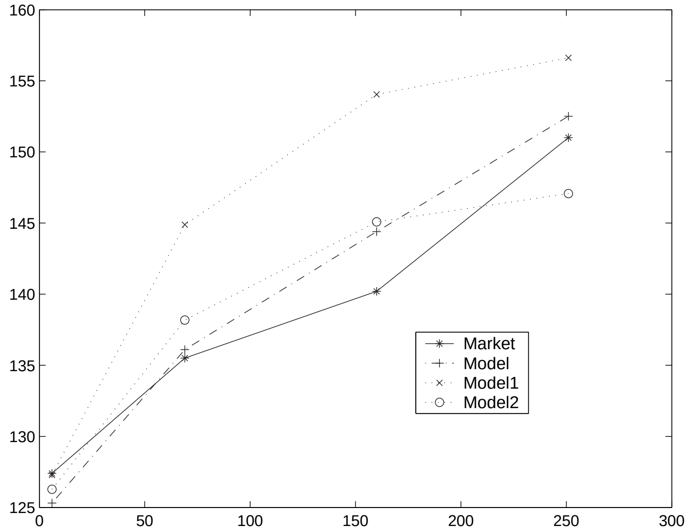

Figure 3: Model v.s. market. Model price corresponds to the fitted model with time varying mean-reverting level $\theta(t)$. Model1 and Model2 corresponds to constant mean-reverting level of 0.027 and 0.024, respectively. Mean Squared Error is calculated for each model with respect to market price. The pricing error of the constant mean reversion models cannot be reduced due to the rigidity of corresponding variance term structure.

Appendix

A Proof of Proposition 1, 2 and 3

With instantaneous variance given by equation (7), the instantaneous forward variance at time T, $V_T(0) = E_0(V_T)$. By taking expectation of (7),we have

$$E_0(V_T) = e^{-\kappa T}V_0 + \kappa \int_0^T e^{-\kappa(T-t)}\theta(t)dt \tag{27}$$

Multiply the above by $e^{\kappa T}$ and differentiate by T, we get the result of Proposition 1.

To prove Proposition 3, integrate equation (27)with respect to T, using integration by part, we get the result for A and B. VIX futures pricing formula follows directly from definition.

To prove Proposition 2, with instantaneous variance given by equation (7),define

$$P(V_t, t) = E_t^Q\left[e^{uV_T}|V_t\right] \tag{28}$$

which satisfies the following backward PDE

$$\frac{\partial P}{\partial t} + \kappa(\theta(t) - V)\frac{\partial P}{\partial V} + \frac{1}{2}\sigma^2 V \frac{\partial^2 P}{\partial V^2} = 0 \tag{29}$$

with the terminal condition

$$P(V, t = T) = e^{uV}. \tag{30}$$

Postulate a solution for P as $P(V,t) = e^{\alpha(t,u)+\beta(t,u)V}$. Substitute into (29), and arrange terms, we get the following ODE:

$$\dot{\beta}(t,u) = \kappa\beta(t,u) - \frac{1}{2}\sigma^2\beta(t,u) \tag{31}$$

$$\dot{\alpha}(t,u) = -\kappa\theta(t)\beta(t,u) \tag{32}$$

with the initial (terminal) condition $\beta(T,u) = u, \alpha(T,u) = 0$. Solving for the above ODE we get the characteristic function of probability density of V_t.

To prove the existence condition (22), observe that when $\theta(t)$ is constant, we get the solution for α and β as follows:

$$\beta(t,u) = \frac{\kappa u e^{-\kappa(T-t)}}{\kappa - \frac{1}{2}\sigma^2 u(1 - e^{-\kappa(T-t)})} \tag{33}$$

$$\alpha(t,u) = -\frac{2}{\sigma^2}\kappa\theta \ln\left[1 - \frac{\sigma^2 u}{2\kappa}(1 - e^{-\kappa(T-t)})\right] \tag{34}$$

With the condition defined by (22), we have

$$\lim_{\phi\to\infty} \beta(t, i\phi) = -C_0 \tag{35}$$

$$\lim_{\phi\to\infty} |\alpha(t, i\phi)| = \ln(C_1 i\phi)^{-\frac{2\kappa\theta}{\sigma^2}} \tag{36}$$

where C_0 and C_1 are positive real constants.

Therefore, when $\theta(t)$ a time dependent deterministic function, there exists a constant C_2 such that (20) can be approximated as

$$|\alpha(t, i\phi)| < C_2 + \kappa\bar{\theta}\int_t^T \beta(s, i\phi)ds \tag{37}$$

Hence, we have

$$|\alpha(t, i\phi)| \leq \ln(C_3 i\phi)^{-\frac{2\kappa\bar{\theta}}{\sigma^2}} \tag{38}$$

for some constant C_3. We proved the existence condition (22).

B MLE and Probability Density Function

Let $x_t = \ln(S_t)$, from Ito's Lemma we have

$$dx_t = (\mu - \frac{1}{2}V_t)dt + \sqrt{V_t}dB_1(t), \tag{39}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V\sqrt{V_t}dB_2(t), \tag{40}$$

with $E\left[dB_1(t)dB_2(t)\right] = \rho dt$. $(Z_1(t), Z_2(t))$as standard Brownian Motion, we can write

$$dB_1(t) = \sqrt{1-\rho^2}dZ_1(t) + \rho dZ_2(t), \qquad dB_2(t) = dZ_2(t), \tag{41}$$

Substitute (41) into (40), we get $\sqrt{V_t}dZ_2(t) = \frac{1}{\sigma_V}(dV_t - \kappa(\theta - V_t)dt)$ and substitute into (39), we have

$$dx(t) = (\mu - \frac{1}{2}V_t)dt + \frac{\rho}{\sigma_V}(dV_t - \kappa(\theta - V_t)dt) + \sqrt{1-\rho^2}\sqrt{V_t}dZ_1(t) \tag{42}$$

We wish to evaluate the transition density $P\left[(x_\delta, V_\delta)|(x_0, V_0)\right]$,where δ is the time between consecutive observations. We take advantage of Bayes' Rule, and the fact that V_t is itself a markov process, to obtain

$$P\left[(x_\delta, V_\delta)|(x_0, V_0)\right] = P\left[x_\delta|x_0, V_0, V_\delta\right]P\left[V_\delta|V_0\right] \tag{43}$$

The conditional distribution of V_t given V_0 is a noncentral chi-square with density given by

$$p(V_\delta|V_0) = ce^{c(V_\delta + e^{-\kappa\delta}V_0)}(\frac{V_\delta}{e^{-\kappa\delta}V_0})^{q/2}I_q(2c(V_\delta V_0 e^{-\kappa\delta})^{\frac{1}{2}}), \tag{44}$$

where $c = 2\kappa(1 - e^{-\kappa\delta})^{-1}, q = 2\kappa\theta - 1$, and I_q denotes the modified Bessel function of the first kind of order q.

There is no known explicit expression for $p(x_\delta|x_0, V_0, V_\delta)$. We base an approximation on the following observation: The distribution of X_δ conditional on X_0 and the entire path of V_t from time 0 to time δ has a known normal density

$$p(x_\delta|x_0, V_s, s \in [0, \delta]) = \phi(x_\delta, m_\delta, \bar{V}_\delta) \tag{45}$$

where $\phi(\cdot, a, V)$ is the density of a normal random variable with mean a and variance V, and

$$m_\delta = \int_0^\delta (\mu - \frac{1}{2}V_t)dt + \frac{\rho}{\sigma_V}\int_0^\delta dV_t - \frac{\rho}{\sigma_V}\int_0^\delta \kappa(\theta - V_t)dt + x_0 \tag{46}$$

$$\bar{V}_\delta = (1 - \rho^2)\int_0^\delta V_t dt \tag{47}$$

By the law of iterated expectations,

$$p(x_\delta|x_0, V_0, V_\delta) = E\left[p(x_\delta|x_0, V_s, s \in [0, \delta])|x_0, V_0, V_\delta\right] = E\left[\phi(x_\delta, m_\delta, \bar{v}_\delta)\right], \tag{48}$$

To complete the specification of the conditional density function of the state variables amounts to approximating the expectation in (48). It has been shown in ([10]) that one can approximate $p(x_\delta|x_0, V_0, V_\delta)$ as the conditional density of x_δ given V_s, evaluated at an outcome of the path of V_s that is linear between V_0 and V_δ. This approximation is tractable and accurate for our application.

C MLE Estimation Results

The ML estimation result is as follows:

	κ	θ	σ_V	λ	ρ	μ
Estimate	5.2952	0.0299	0.3837	-12.0644	-0.6413	0.0370
Stddev	0.4424	0.0025	0.0067	0.6644	0.0085	0.0220

The risk premium is strongly negative, while the stock index return μ is not significantly different from zero. This is because most of the return has been explained by the movement correlated with volatility process. The strongly negative risk premium is due to the short term nature of the variance swap rate of VIX. This is a well documented fact that short term skewness of option prices cannot be adequately explained by diffusive volatility alone. For example, adding jumps will reduce greatly the stochastic volatility risk premium. In our WMC application, however, risk premium is determined in a non-parametric way by incorporating all the input information of options data. Therefore, only the physical parameters are used.

D Options Data

We present the options data we used for WMC calibration:

References

[1] Avellaneda, Marco, B. Robert, C. Freidman, N. Grandchamp, L. Kruk, and J. Newman, 2000, Weighted Monte Carlo: A new technique for calibrating asset-pricing models. *International Journal of Theoretical and Applied Finance*, 4(1), 91-119.

[2] Bakshi, S. Gurdip and Nikunj Kapadia, 2003, Delta Hedged Gains and the Negative Volatility Risk Premium, *Review of Financial Studies* 16, 527-566.

[3] Bakshi, S. Gurdip, Charles Cao and Zhiwu Chen, 1997, Empirical Performance of Alternative Option Pricing Models, Journal of Finance 52(5), 2003-2049.

[4] Breeden, D. and R. Litzenberger, 1978, Prices of state-contingent claims implicit in option prices, *Journal of Business*, 51, 621-651.

[5] Carr, Peter, and Liuren Wu, 2003, A Tale of two indices, Working paper, Bloomberg L. P. and City University of New York.

[6] Carr, Peter, and Liuren Wu, 2004, Variance Risk Pemia, Working paper, Bloomberg L.P. and City University of New York.

[7] Chicago Board Options Exchange, 2003, VIX-CBOE Volatility Index, http://www.cboe.com/micro/vix/vixwhite.pdf.

[8] Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, 1985, A theory of the term structure models, *Econometrica* 53, 385-407.

[9] Demeterfi, Kresimir, Emanual Derman, Micheal Kamal, and Joseph Zou, 1999, More than you ever wanted to know about volatility swaps. *Goldman Sachs quanititative research notes.*

[10] Duffie, Darrell, Lasse Hehe Pedersen, and Kenneth J. Singleton, 2003, Modeling sovereign yield spreads: a case study of Russian debt, *Journal of Finance*, 58(1), 119-159.

[11] Dupire, Bruno, 1993, Arbitrage pricing with stochastic volatility, *Working paper*.

[12] French, K. R., G. W. Schwert, and R. F. Stambaugh, 1987, Expected stock returns and volatility, *Journal of Financial Economics*, 19, 3-29.

[13] Heston, Stephen, 1993, Closed-form solution for options with stochastic volatility, with application to bond and currency options, *Review of Financial Studies* 6, 327-343.

[14] Howison, Sam, Avraam Rafailidis, and Henrik Rasmussen, 2004, On the pricing and hedging of volatility derivatives, *Applied Mathematical Finance*, 11, 317-346 (2004).

[15] Hull, John, and Alan White, 1990, Pricing interest-rate derivative securities, *Review of financial studies*, 3, 573-592.

[16] Whaley, Robert E., 1993, Deriavtives on market volatility: Hedging tools long overdue, *Journal of Derivatives* 1, 71-84.

[17] Zhang, Jin E., and Yingzi Zhu, 2005, VIX futures, Working paper, The University of Hong Kong and Tsinghua University.

[18] Zhu, Yingzi, and Marco Avellaneda, 1998, A risk-neutral volatility model, *International Journal of Theoretical and Applied Finance*, 1(2), 289-310.

[19] Zhu, Yingzi, and Marco Avellaneda, 1997, An E-ARCH model for the term structure of implied volatility of FX options, *Applied Mathematical Finance*, 4, 81-100 (1997).

Exp (Days)	Strike	Type	Price	Volume
37	1220	Call	12.5000	1353
37	1225	Call	9.8000	1928
37	1230	Call	8.0000	1176
37	1250	Call	3.1000	2307
37	1275	Call	0.9500	4557
37	1120	Put	1.7500	1504
37	1125	Put	1.8000	1627
37	1150	Put	3.4000	4456
37	1170	Put	7.1000	1257
37	1175	Put	6.2000	2028
37	1180	Put	7.8000	1098
37	1200	Put	12.5000	6524
72	1225	Call	17.0000	1131
72	1275	Call	3.8000	1039
72	1050	Put	1.5000	1000
72	1150	Put	8.0000	1020
100	1215	Call	29.0000	3730
100	1100	Put	5.5000	1274
100	1150	Put	12.2000	1554
100	1175	Put	16.9000	1370
100	1200	Put	24.8000	1281
100	1215	Put	30.0000	4239
191	1325	Call	7.0000	2502
191	1075	Put	10.6000	4201
191	1100	Put	13.2000	1370
191	1125	Put	17.4000	1050
282	1250	Call	39.7000	1275
282	1350	Call	10.0000	1255
282	750	Put	0.9000	2640
282	850	Put	2.2500	20000
282	1025	Put	9.8000	1200
282	1050	Put	12.5000	1000
282	1075	Put	15.5000	1750
282	1175	Put	35.0000	2100
282	1200	Put	43.2000	1268

Table 2: The market options data as selected by the criteria described in the paper, are given as follows. The S&P 500 spot market $S_0 = 1209.3$.